Virtual Seifert Surfaces, Almost Classical Knots, and Alexander Polynomials of Boundary Links in $\mathbb{S}^3$

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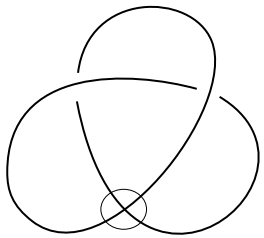
Virtual Knots

The four pictures below all represent the same virtual knot.



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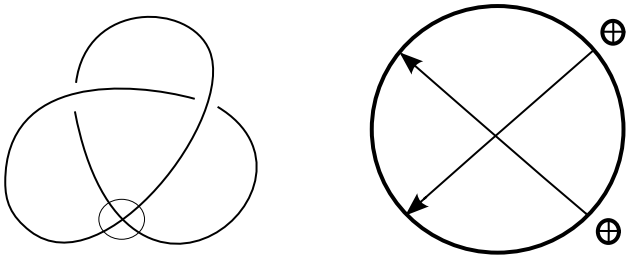

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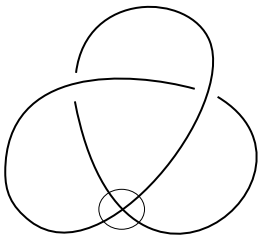

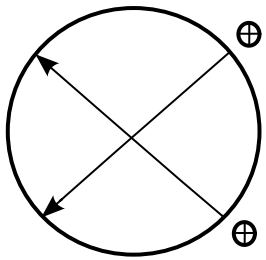

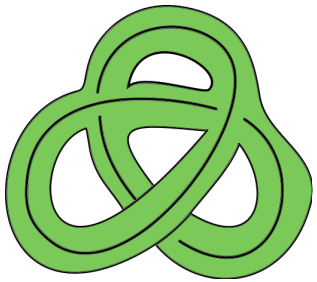

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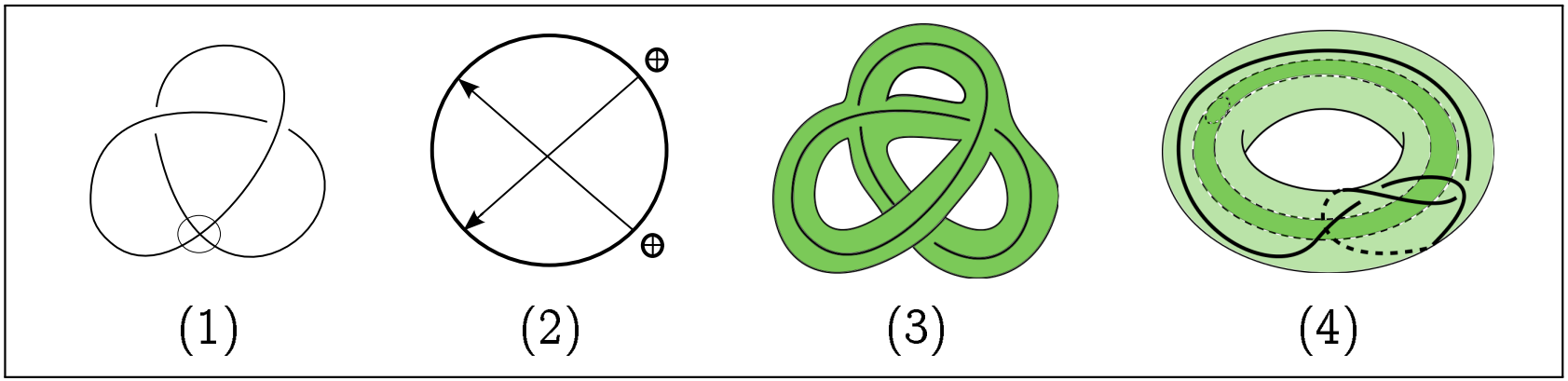

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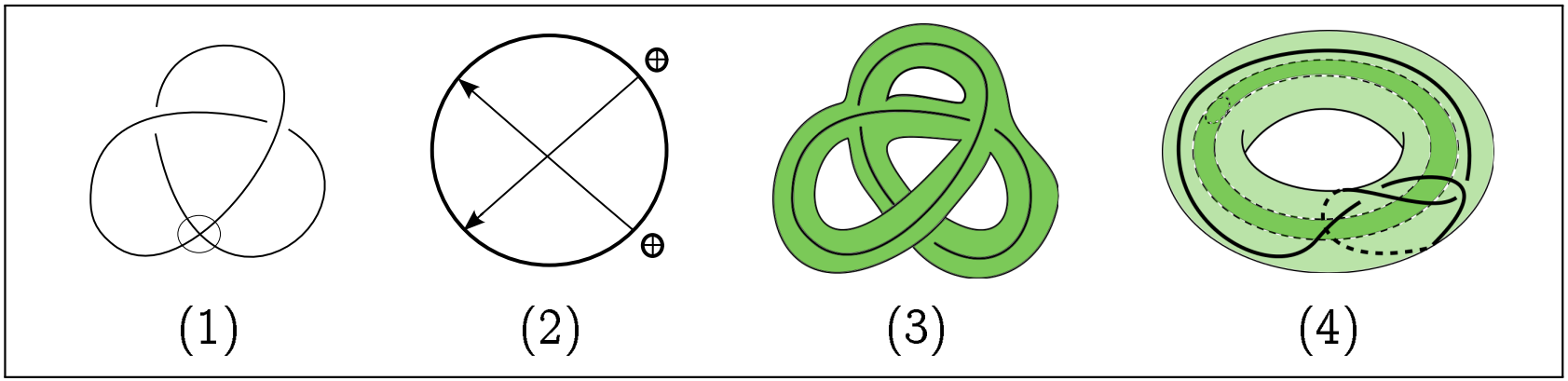


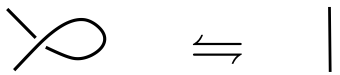
A virtual knot is a classical knot if and only if it can be represented as a diagram on a planar surface.

Extended Reidemeister Moves:

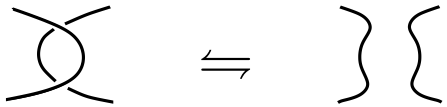
Definition (Kauffman '99)

Two virtual knot diagrams are *equivalent* if a finite sequence of *extended Reidemeister moves* takes one to the other.

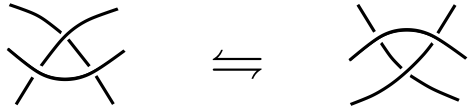
R1:



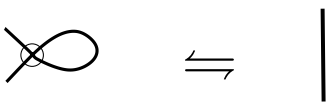
R2:



R3:



V1:



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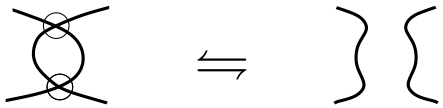

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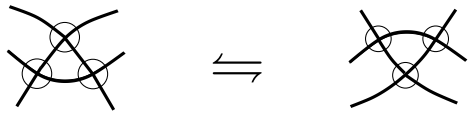
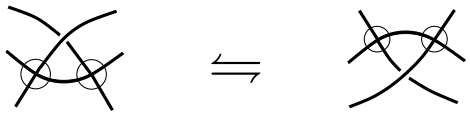


V4:



Viirtual Knot Invariants

Because of the diagrammatic definition, virtual knot invariants are typically defined combinatorially:

- Writhe Polynomial
- Three Loop Invariant
- Goussarov-Polyak-Viro finite-type invariants
- Transcendental polynomial
- Generalized Alexander polynomial
- Jones Polynomial
- Khovanov Homology
- Rasmussen Invariant
- Arrow Polynomial
- Parity Bracket
- Parity Biquandles
- Long Quaternionic Polynomial
- Graded Genus
- Signatures (via Goeritz matrices)
- And many more!

General Problem

Problem: Classical knots embed in virtual knots, but virtual knot invariants are either trivial on classical knots or extend known invariants of classical knots.

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Here we will show how virtual knot invariants can be found hiding inside invariants of classical links. Specifically, the Alexander polynomial of almost classical knots is a specialization of the multi-variable Alexander of certain boundary links.

Definition (Alexander Quandles)

Let A be a module over the ring $\Lambda = \mathbb{Z}[t^{\pm 1}]$. The *Alexander quandle* on A is given by the operation:

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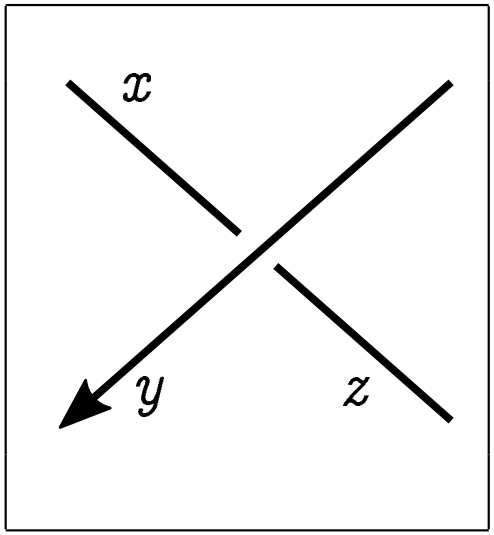
- For each crossing, there is a relation $x_i \triangleright x_j = x_k$.
- $z = x \triangleright y = tx + (1 - t)y$



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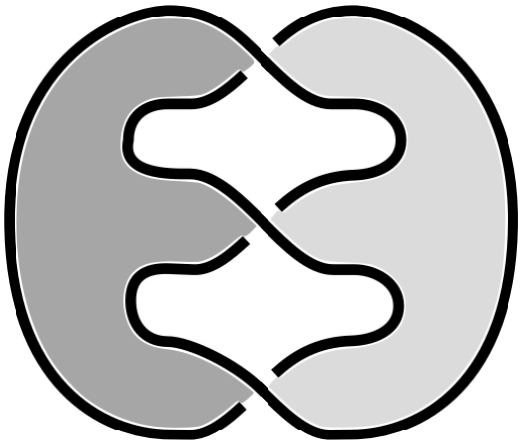
- The *k-th Alexander polynomial* $\Delta^k(t)$ is the g.c.d of the minors generating I_k. It is well-defined up to multiplication by $t^{\pm N}$.

Seifert Surface Definition



Oriented Surface

$\partial F = K$

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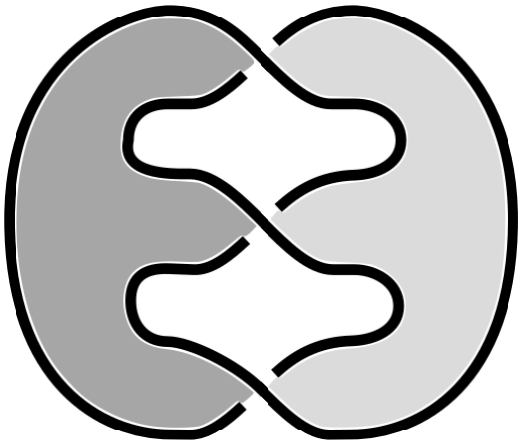

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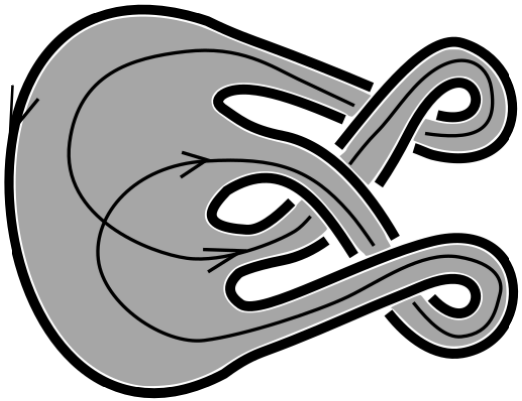

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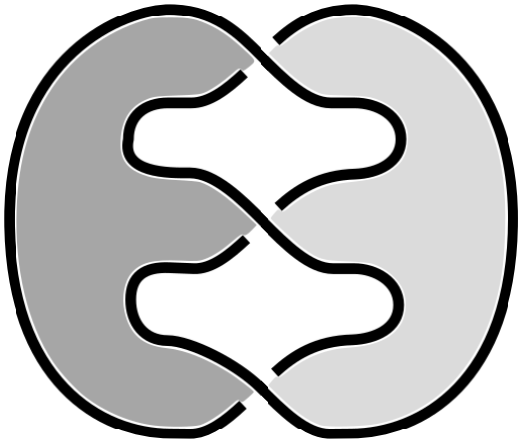

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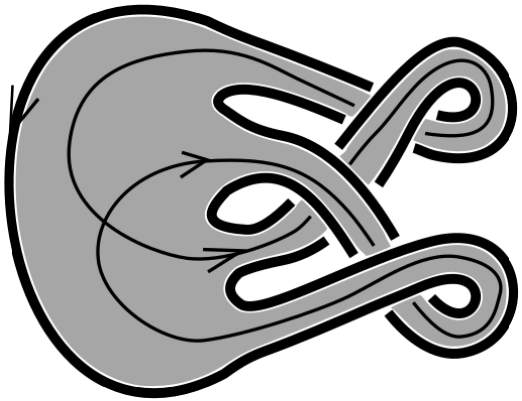

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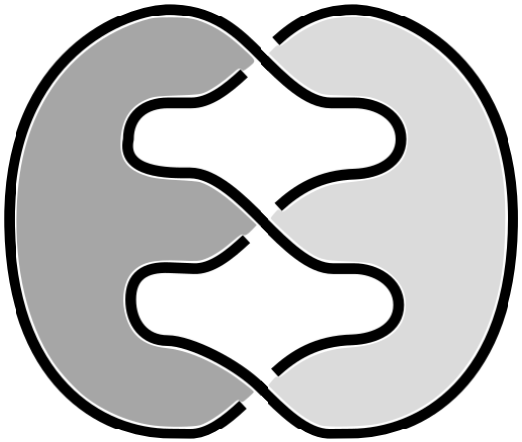

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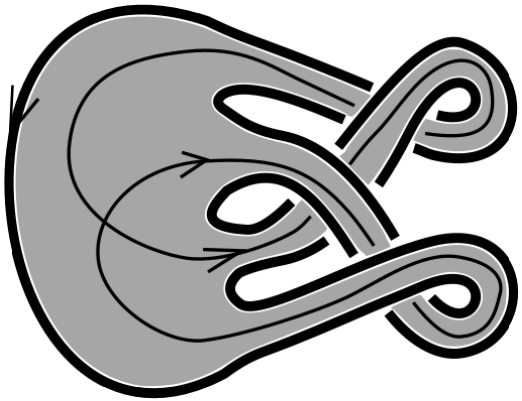

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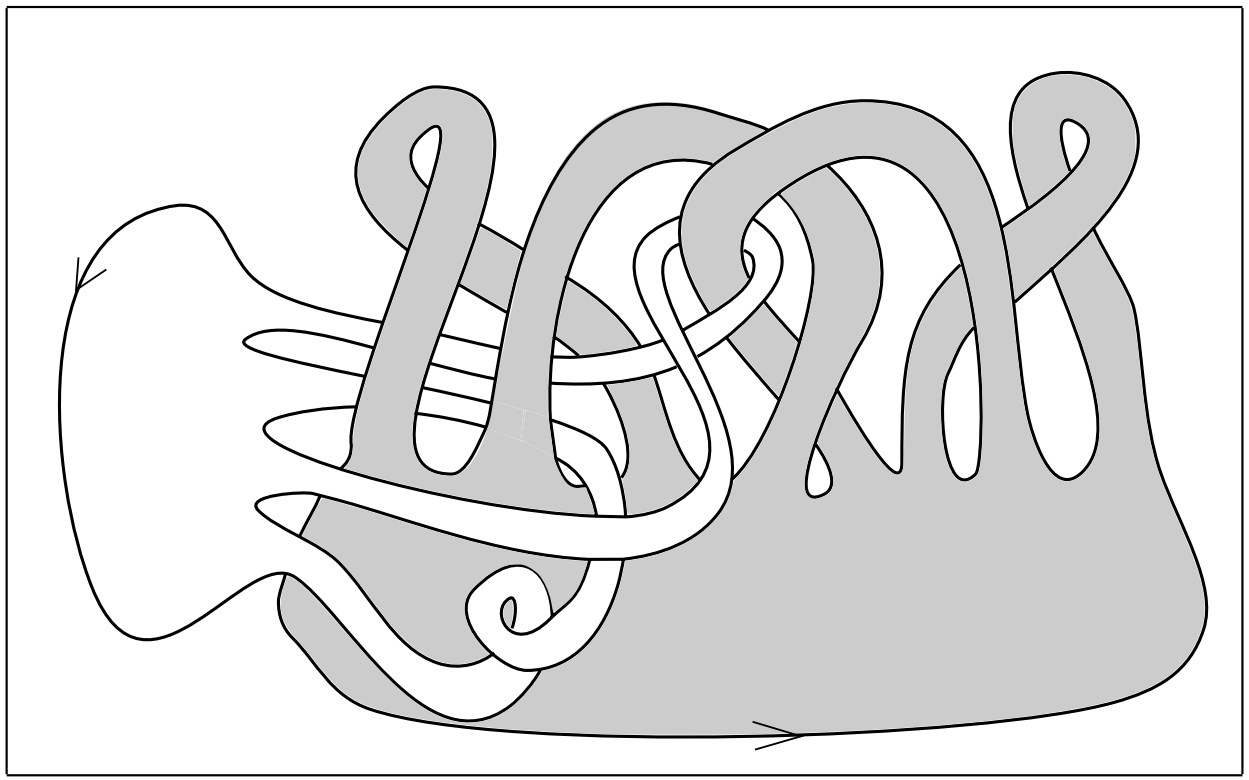
Alexander Polynomial :
$\Delta_K = \det(tA - A^\tau)$
$(\Delta_K = \Delta^1)$

Definition (Boundary Link)

An oriented n-component link $L = J_1 \sqcup \cdots \sqcup J_n$ is said to be a *boundary link* if each J_i bounds an oriented Seifert surface Σ_i such that $\{F_i\}$ satisfies $F_i \cap F_j = \emptyset$ for all $i \neq j$.



Multi-Variable Alexander Polynomial (MVAP):

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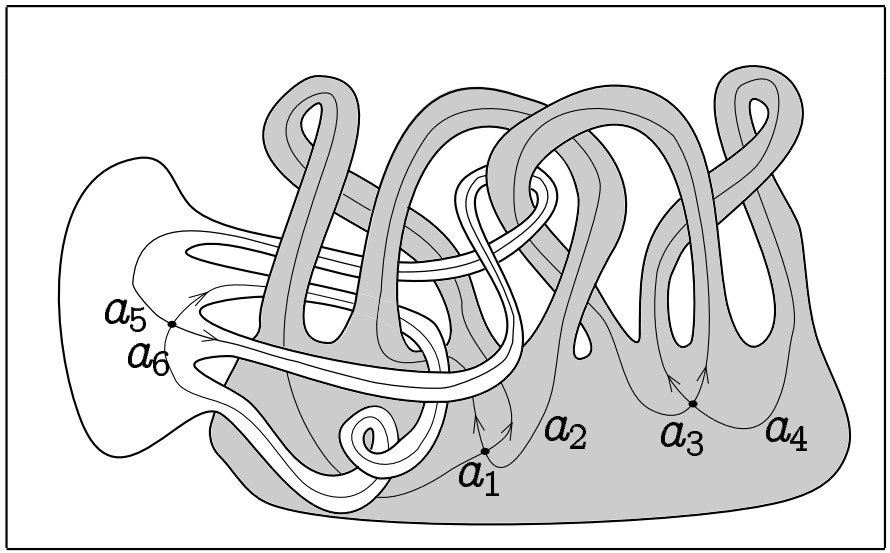

Example: For L on right:

$$\begin{aligned}
\nabla_L(t_1,t_2) &= -2 + 8t_1 - 10t_1^2 + 6t_1^3 - t_1^4 + 2t_2 - 10t_1t_2 + 15t_1^2t_2 - 10t_1^3t_2 \\
&+ 2t_1^4t_2 - t_2^2 + 6t_1t_2^2 - 10t_1^2t_2^2 + 8t_1^3t_2^2 - 2t_1^4t_2^2.
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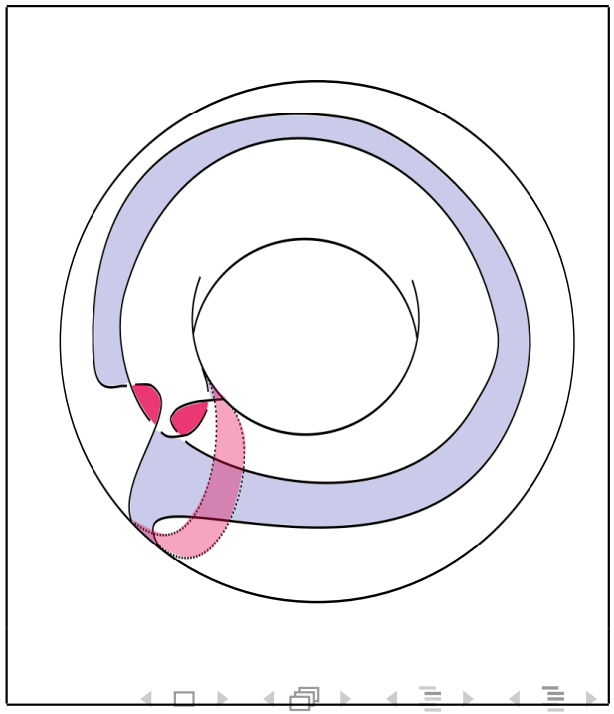
Almost Classical Knots

Definition (Silver-Williams '06)

An *almost classical knot* is a virtual knot possessing an Alexander numerable knot diagram.

- $\iff$ bounds a Seifert surface in some $\Sigma \times \mathbb{I}$ representation.
- $\iff$ is homologically trivial in some $\Sigma \times \mathbb{I}$ representation.
- $\iff$ has a diagram s.t. index of every crossing is 0.



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- directed Seifert matrix: $V^{\pm} := (\mathrm{lk}_\Sigma(\alpha_i^{\pm}, \alpha_j))$.
- $\Delta_K(t) = \det(t \cdot V^- - V^+)$ (Boden et al '16)

Definition of Virtual Covers

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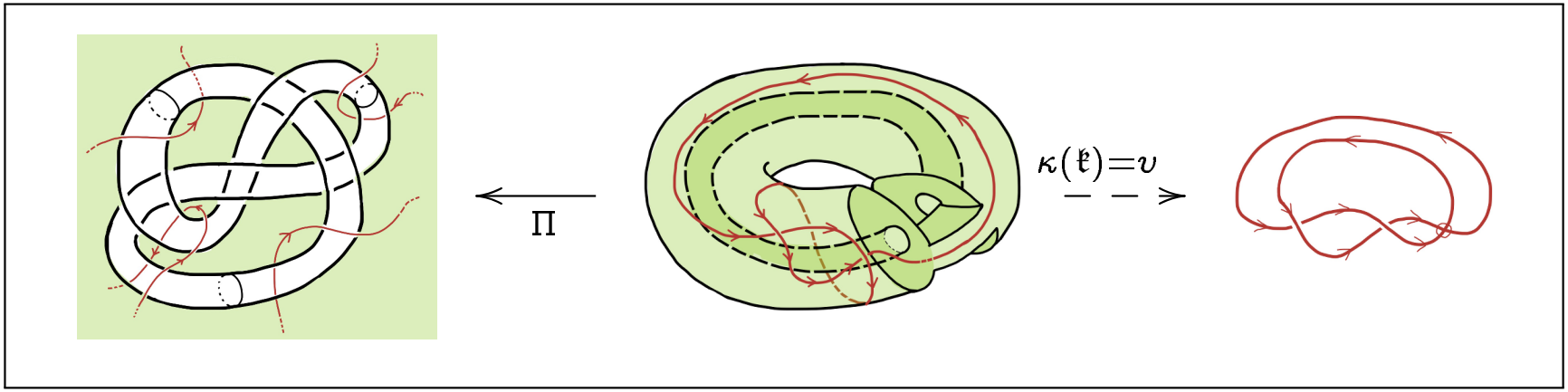
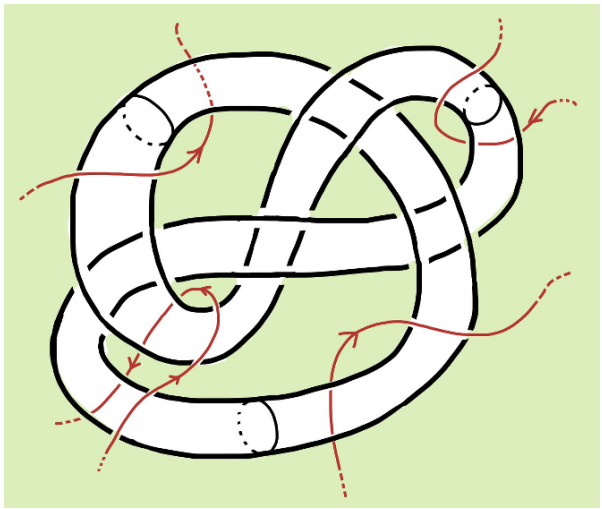
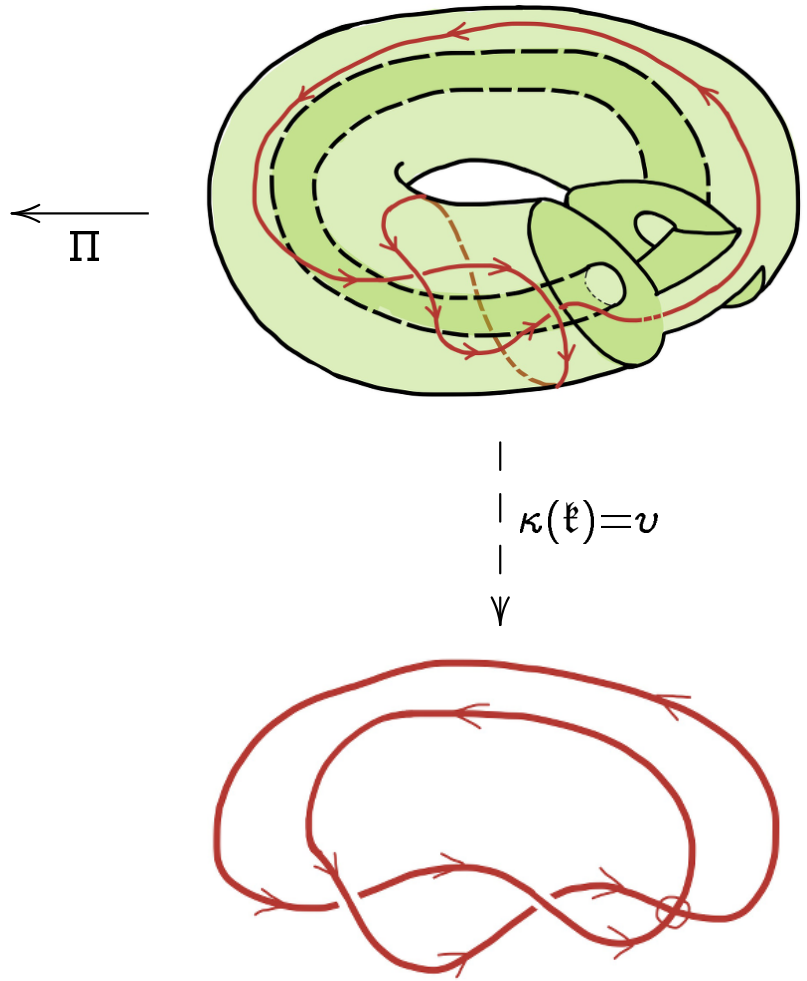
Definition (Virtual Cover, Associated Virtual Knot)

K is a knot in N s.t. $\exists$ a knot $\mathfrak{k}$ in$\Sigma \times \mathbb{R}$ with $\Pi(\mathfrak{k}) = K$. A *virtual cover* of K is choice $\mathfrak{k}$ of lift by Π for K. The stabilization of $\mathfrak{k}$ to a virtual knot v is called the *associated virtual knot*.



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Fibered Links

Let $N_J = \mathbb{S}^3 \setminus J$ be the exterior of a link J.

Definition (Fibered Link)

A link J in $\mathbb{S}^3$ is said to be *fibered* if it has a Seifert surface Σ_J such that $(\mathbb{S}^3 \setminus \Sigma_J, (\mathbb{S}^3 \setminus \Sigma_J) \cap \partial N_J)$ and $(\Sigma_J \cap N_J, \partial(\Sigma_J \cap N_J) \times \mathbb{I}$ are diffeomorphic as pairs.

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- Cutting out a Seifert surface gives a thickened surface $\Sigma_J \times [0, 1]$.
- The surface Σ_J is called a *fiber* of J.
- A fibered knot admits an o.p. covering space $\Pi : \Sigma_J \times \mathbb{R} \to N_J$.
- Examples: Trefoil, figure eight are genus 1 fibered knots. 8_{21} is a genus 2 fibered knot.

Definition (Special Seifert Form (SSF))

An link $L = J \sqcup K$ with J a fibered n component link and K a knot is said to be in special Seifert form if K disjoint from a fiber Σ_J of J.

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Theorem (C-Manturov ('13), C. ('16))

For $i = 1, 2$, let $L_i = J_i \sqcup K_i$ be oriented links with J_i a fibered link, K_i in SSF with respect to a connected fiber Σ_i of J_i, and v_i the associated virtual knot. If $L_1 \leftrightharpoons L_2$, then $v_1 \leftrightharpoons v_2$.

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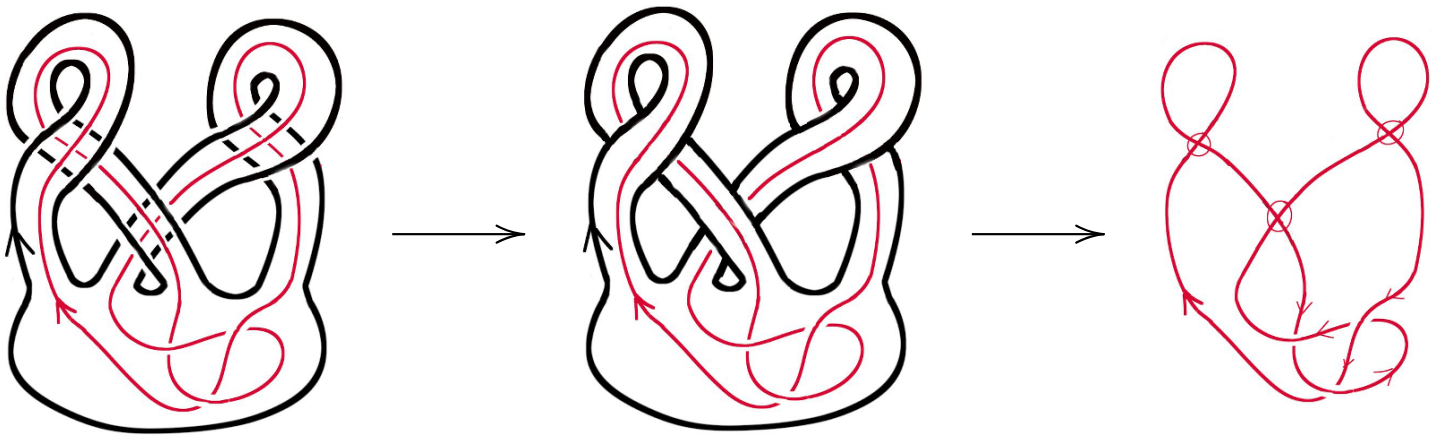


Main Theorem

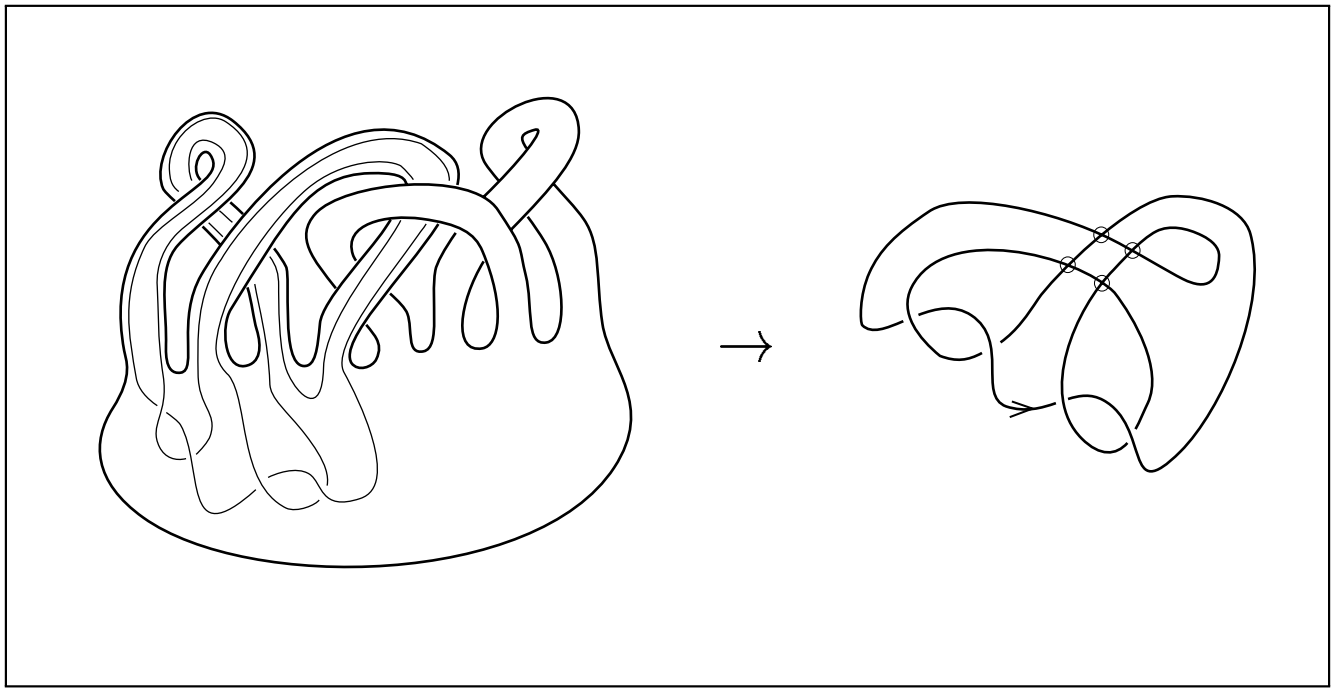
Theorem (C.-Todd)

Let $L = J \sqcup K$ be a boundary link with J fibered. Suppose that K bounds a Seifert surface Σ_K disjoint from a fiber Σ_J of J. Then the associated virtual knot v of L is almost classical. Moreover, each of the following hold:

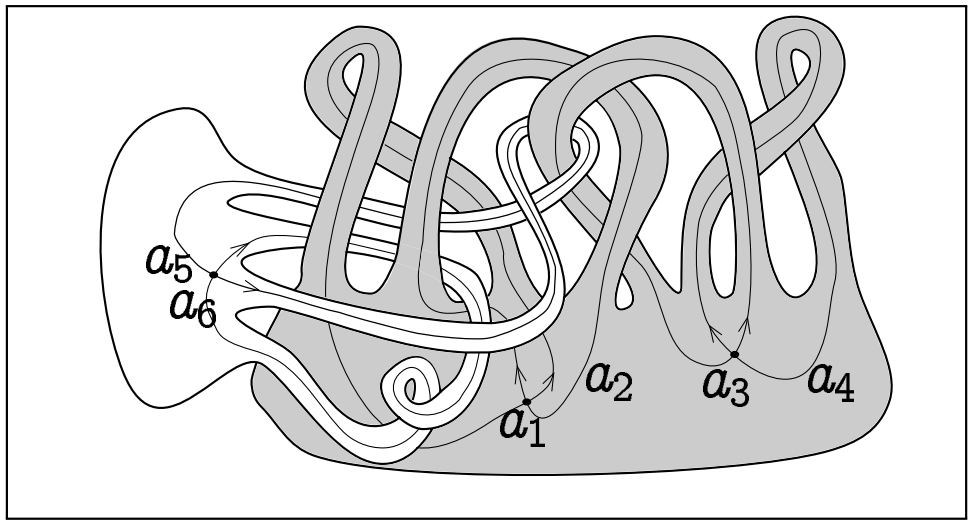
1. $\nabla_L(s, 1) = \Delta_J(s)$,
2. $\nabla_L(1, t) = \Delta_K(t)$, *and*
3. $\nabla_L(0, t^{-1}) \doteq \Delta_v(t)$.

Example:

$L = J \sqcup K$, $J \leftrightharpoons 8_{21}$ (a fibered knot), $v \leftrightharpoons 4.105$ (an AC knot)



Example (cont):



$$\begin{aligned}
\nabla_L(t_1,t_2) &= -2+8t_1-10t_1^2+6t_1^3-t_1^4+2t_2-10t_1t_2+15t_1^2t_2-10t_1^3t_2 \\
&+ 2t_1^4t_2-t_2^2+6t_1t_2^2-10t_1^2t_2^2+8t_1^3t_2^2-2t_1^4t_2^2. \\
\nabla_L(0,t^{-1}) &= -2-\frac{1}{t^2}+\frac{2}{t} \doteq 1-2t+2t^2
\end{aligned}$$

Proof of (3):

- Since K bounds a Seifert surface disjoint from a fiber of J, K bounds a Seifert surface in $\mathbb{S}^3 \setminus \Sigma_J \approx \Sigma_J \times [0,1]$. Thus, K is in SSF and the associated virtual knot is AC.

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Lemma

Suppose that y is a knot in $\mathbb{S}^3 \setminus \Sigma_J$ and that x is a knot in $(\mathbb{S}^3 \setminus \Sigma_J) \setminus y$. Let $\beta = \{a_1, \ldots, a_{2g}\}$ be a symplectic basis for the first homology of Σ_J. For an arbitrary knot z in $\mathbb{S}^3 \setminus \Sigma_J$ let $l_z^\tau = (lk(a_1, z), \ldots, lk(a_g, z)))$. Then:

$$lk_\Sigma(y, x) = lk(y, x) - l_y^\tau A_J^{-1} l_x,$$

where A_J is the standard Seifert matrix for Σ_J with respect to β.

Proof of (3), continued:

The matrix A defining the MVAP of can by given in block form as:

$$A = \left[\begin{array}{c|c} A_J & B \\ \hline B^\tau & A_K \end{array}\right] \tag{1}$$

Proof of (3), continued:

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A Seifert matrix of v is given by:

$$V^- = A_K - B^\tau A_J^{-1} B \text{ and } V^+ = A_K^\tau - B^\tau A_J^{-1} B.$$

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$$\det(AT - A^\tau) = \left|\begin{array}{c|c} t_1 A_J - A_J^\tau & (t_2-1)B \\ \hline (t_1-1)B^\tau & t_2 A_K - A_K^\tau \end{array}\right|.$$

Proof of (3), continued:

Plug in 0 for t_1. Then multiply this matrix by a matrix of determinant 1.

$$\left[\begin{array}{c|c} I & O \\ \hline -B^t(A_J^\tau)^{-1} & I \end{array}\right] \cdot \left[\begin{array}{c|c} -A_J^\tau & (t_2-1)B \\ \hline -B^\tau & t_2A_K - A_K^\tau \end{array}\right]$$

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$$((t_2A_K - A_K^\tau) - (t_2-1)B^\tau(A_J^\tau)^{-1}B)^\tau = t_2A_K^\tau - A_K - (t_2-1)B^\tau A_J^{-1}B.$$

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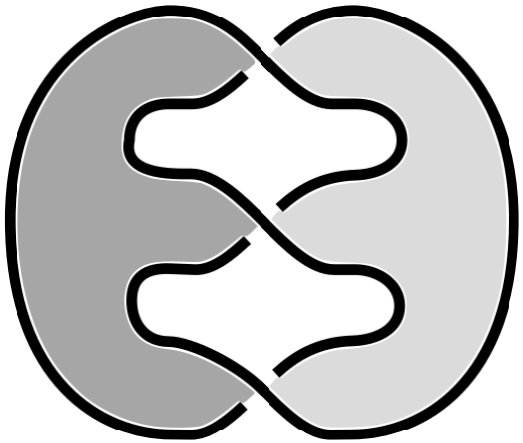
Now perform the substitution $t_2 = s^{-1}$ to obtain:

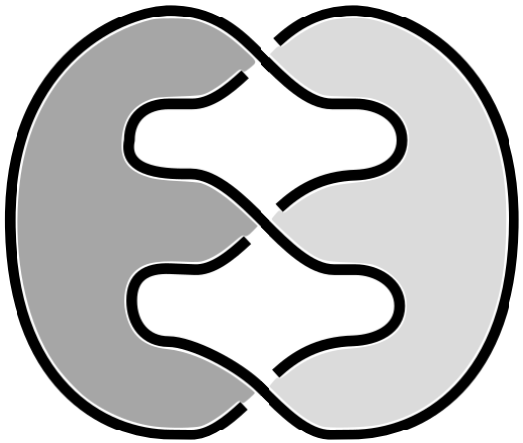
$$\begin{aligned} s^{-1}A_K^\tau - A_K - (s^{-1}-1)B^\tau A_J^{-1}B &\doteq (-s)(s^{-1}A_K^\tau - A_K - (s^{-1}-1)B^\tau A \\ &= sA_K - A_K^\tau - (s-1)B^\tau A_J^{-1}B \\ &= sV^- - V^+. \end{aligned}$$

Classical Seifert surfaces



Three twisted bands.
(canonical)

Classical Seifert surfaces



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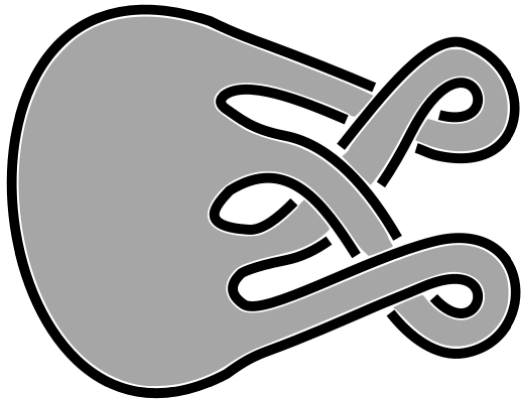
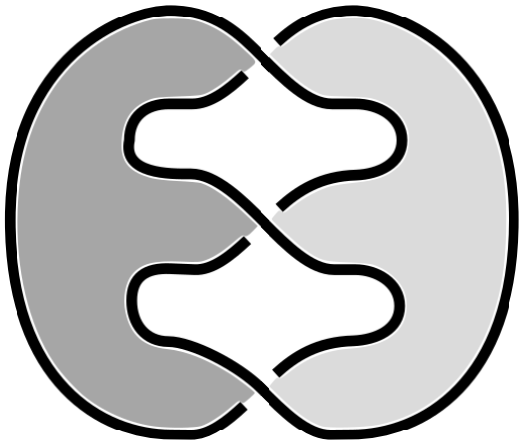


Ideally: No twists!.
(band projection)

Classical Seifert surfaces



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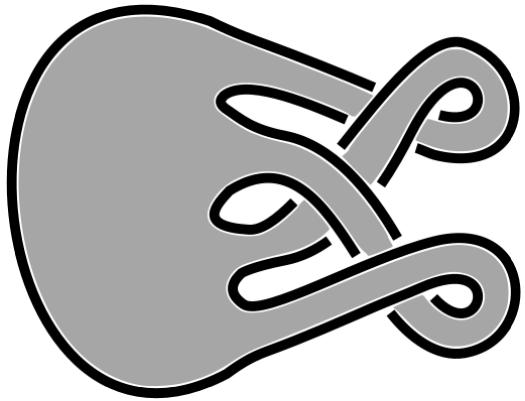

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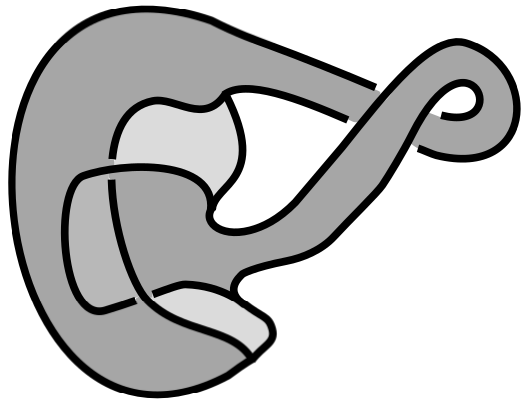
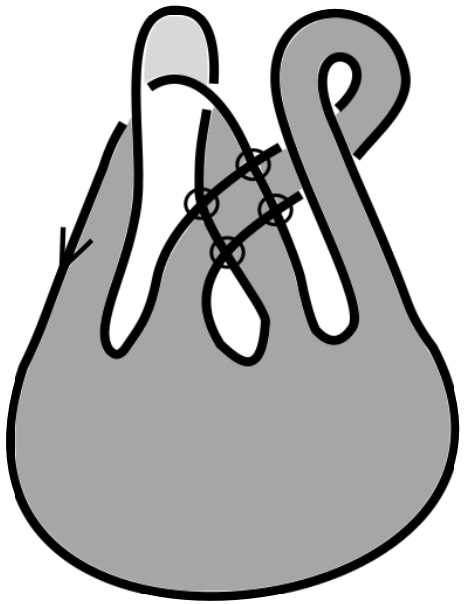


Two twisted bands.
(typical intermediate)

Virtual Seifert surfaces



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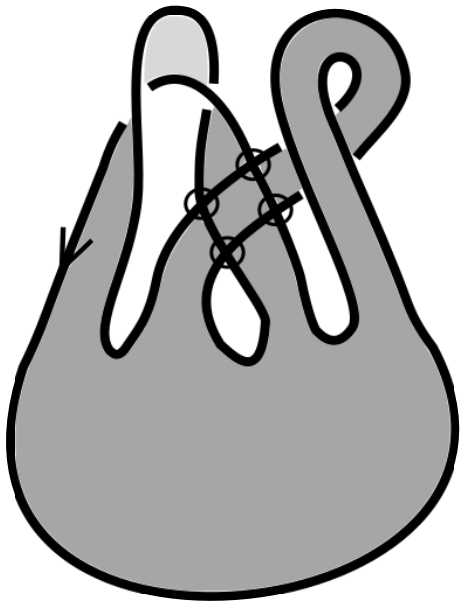

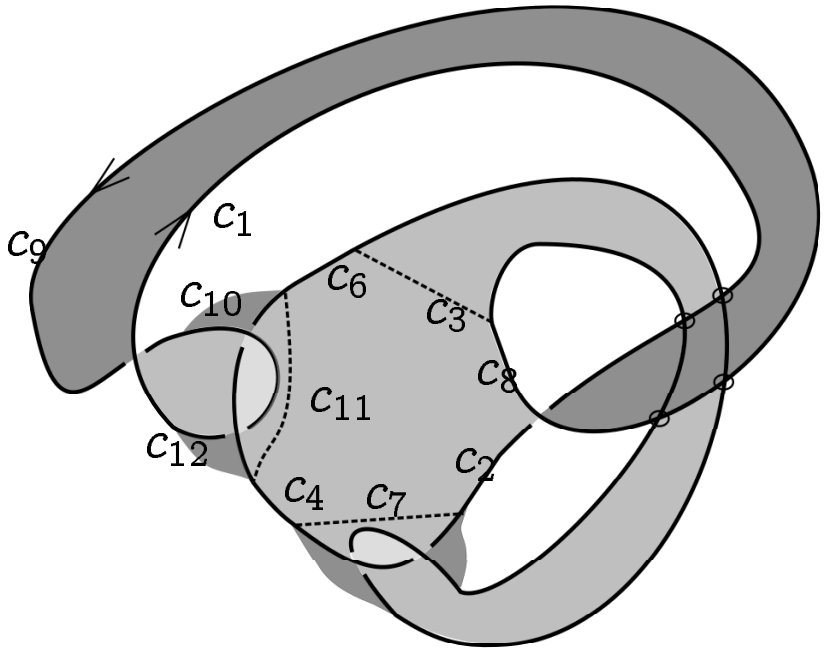

Existence of virtual Seifert surfaces

Theorem (Boden-C.-Gaudreau)

For every AC Gauss diagram D, there is an AC knot diagram K with Gauss diagram D and a canonical virtual Seifert surface of K such that the boundaries of the subsurfaces of S are mapped to the Seifert circles of K. Moreover, every AC knot has a virtual Seifert surface that is a band projection.

Virtual Seifert Surface Algorithm: Let K be an AC knot diagram with Gauss diagram D and Carter surface Σ. Let C_0, C_1, C_2 be the standard cellular decomposition of Σ.

1 Compute $\partial_2 : C_2 \to C_1$ and Seifert cycles from D.

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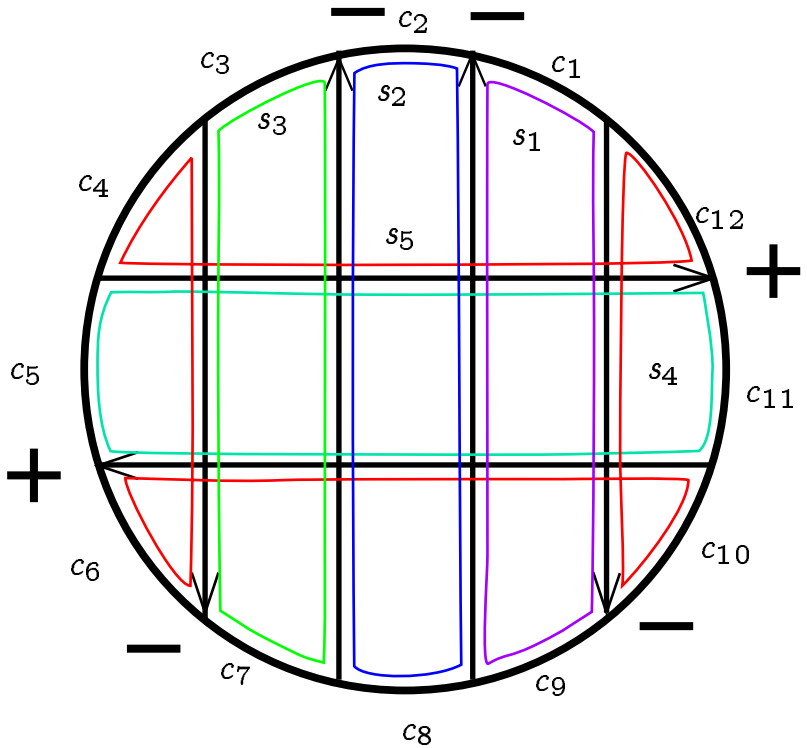
4 Deform to a band projection (through some thickened surface).

Example (6.87548):



$$(\partial_2)^t = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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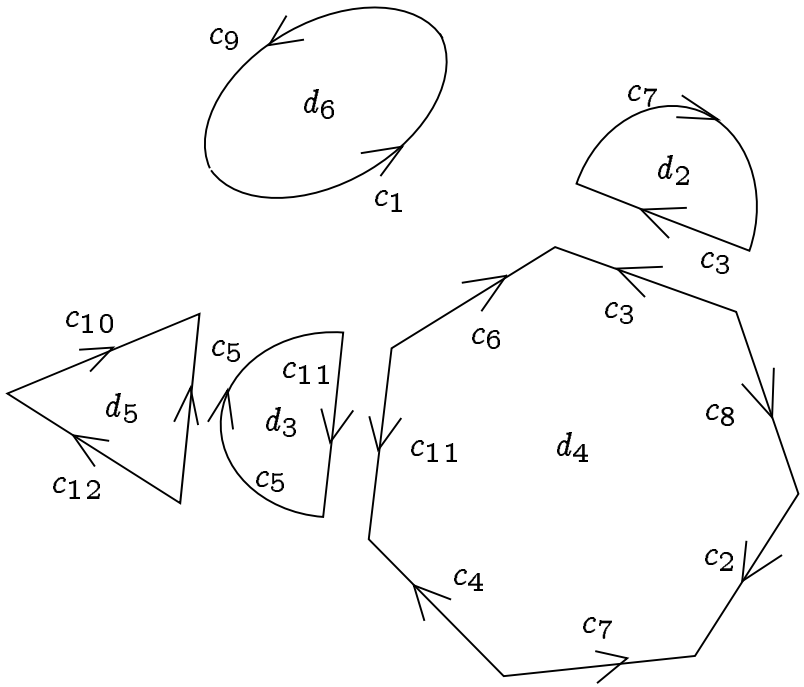

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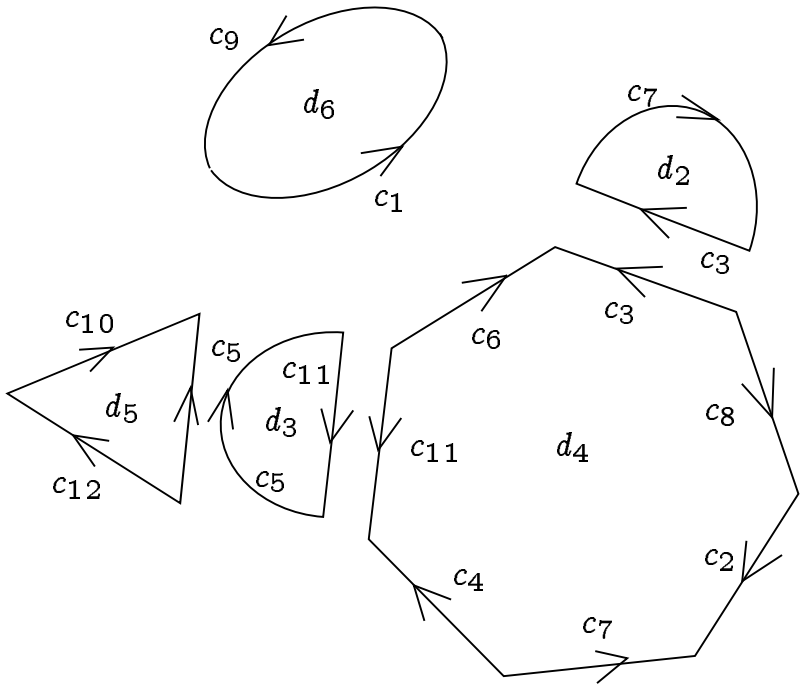
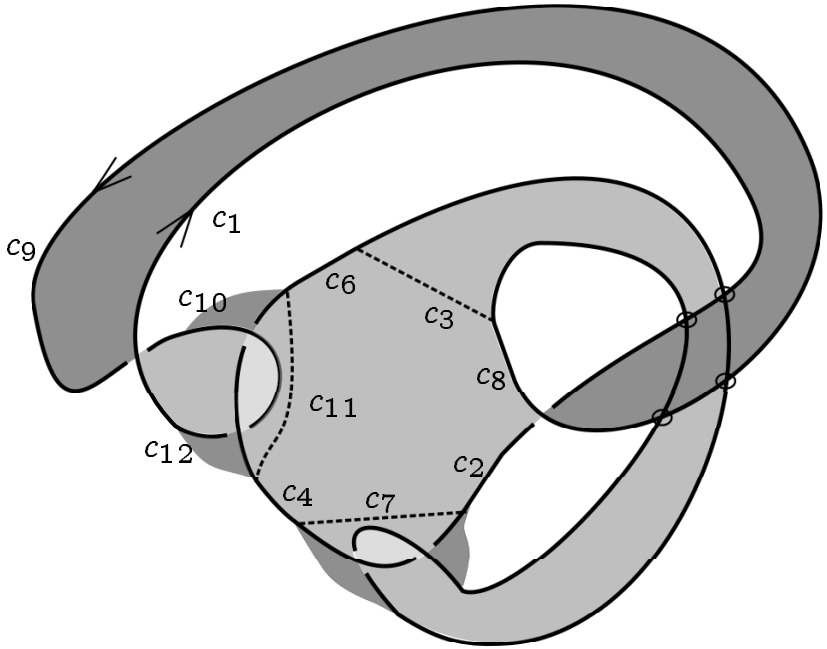


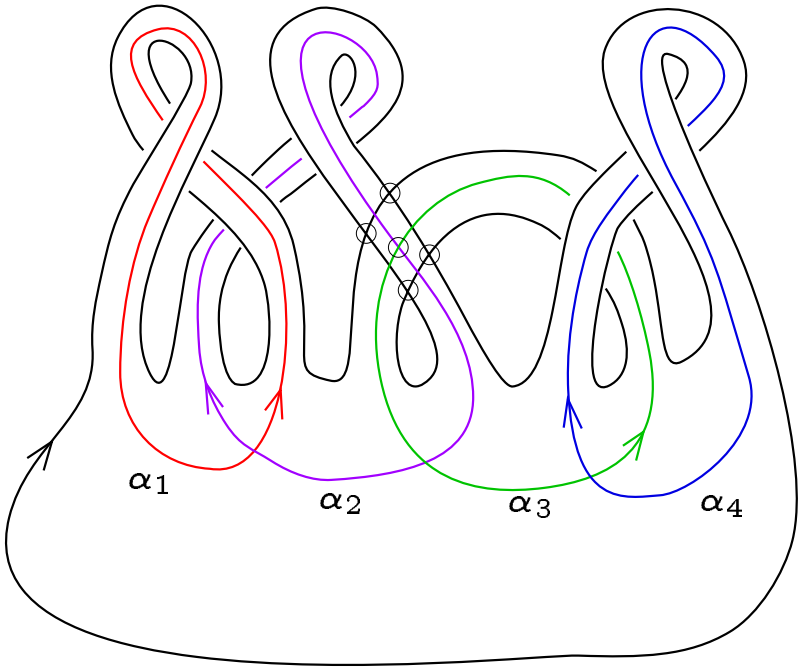


Example (6.87548):

$$V^+ = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V^- = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- $\Delta_v(t) \doteq t^3 - 2t^2 - t + 1.$
- Also can compute slice obstructions.
- 6.87548 is not slice.



Open Questions & Problems

1. Can virtual covers be used to relate quandles of virtual knots to quandles of links in $\mathbb{S}^3$?
2. Use virtual covers to compute Kauffman bracket skein modules (KBSM) for fibered knot complements.
3. Is there a similar formula relating the MVAP of the first elementary ideal to the Alexander polynomial of the zeroth elementary ideal for virtual knots?
4. Can signatures for virtual knots be obtained by specialization of multi-variable signatures function of a multi-component link?

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Virtual knot groups and almost classical knots

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