# Extending Milnor's $\bar{\mu}$-invariants to virtual knots and welded links

**Micah Chrisman**

**Ohio State University**

*CKVK* 5/2020*

# Based on...

Milnor's concordance invariants for knots on surfaces
https://arxiv.org/abs/2002.01505

Virtual concordance and the generalized Alexander polynomial
https://arxiv.org/abs/1903.08737 (w/ H. U. Boden)

# Motivation

# Milnor's $\bar{\mu}$-invariants

## Lower central series

$G$ is a group. $G_1 = G$, $G_2 = [G, G]$, $G_{q+1} = [G_q, G]$.

$$G \triangleright G_2 \triangleright G_3 \cdots$$

$$L \subset S^3, \text{an } m - \text{component link}$$
$$G = \pi_1(S^3 \smallsetminus L)$$
$$F = \langle a_1, \ldots, a_m \rangle$$

## Theorem (Chen-Milnor)

$$G/G_q \cong \langle a_1, \ldots, a_m | [a_1, \lambda_1^{(q)}], \ldots, [a_m, \lambda_m^{(q)}], F_q \rangle$$

# Milnor's $\bar{\mu}$-invariants

## Magnus expansion

$f \in F$, Define $\epsilon(f) \in \mathbb{Z}[[a_1, \ldots, a_m]]$ by:

$$a_i \to 1 + a_i, \quad a_i^{-1} \to 1 - a_i + a_i^2 - a_i^3 + \cdots$$

Let $J = j_1 j_2 \cdots j_r$ be a sequence in $\{1, 2, \ldots, m\}$.

$$\epsilon(f) = 1 + \sum_{J = j_1 j_2 \cdots j_r} \epsilon_J(f) a_{j_1} a_{j_2} \cdots a_{j_r}$$

Applying this to $L \subset S^3$,

$$\mu_{J|k}(L) = \epsilon_J(\lambda_k^{(q)})$$
$$\Delta_J = \gcd\{\mu_{\hat{J}}(L)\},$$

$\hat{J}$ :=delete at least one term from $J$ or any cylic permutation thereof.

# Milnor's $\bar{\mu}$-invariants

## $\bar{\mu}$-invariants

$$\bar{\mu}_J(L) \equiv \mu_J(L) \pmod{\Delta_J}$$

## Properties

- $\bar{\mu}$-invariants are invariants of link concordance (Casson, Stallings).
- $\bar{\mu}$-invariants vanish on boundary links (Smythe).
- $\bar{\mu}$-invariants are trivial on 1-component links i.e. knots.

# Goals & Applications

## GOAL: Construct extended $\bar{\mu}$-invariants

- Invariants of virtual knots and knots in $\Sigma \times I$.
- Defined from LCS of the extended group of a virtual knot.
- Invariants under virtual concordance.
- Vanish on homologically trivial knots in $\Sigma \times I$.

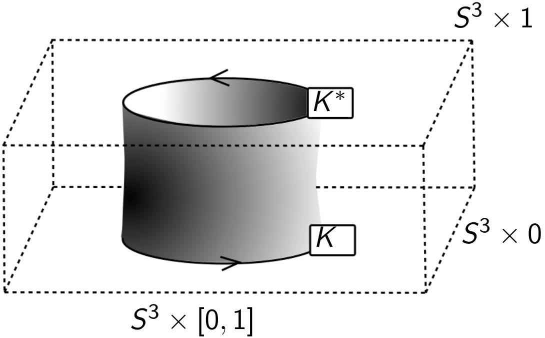
## APPLICATIONS:

1. The virtual knot concordance group is not abelian.
2. Generalize $\bar{\mu}$-invariants to welded links and welded string links.
3. Reduce to 4 (out of 92800) the # of virtual knots from Green's table having unknown slice status.

# Virtual concordance

# Concordance

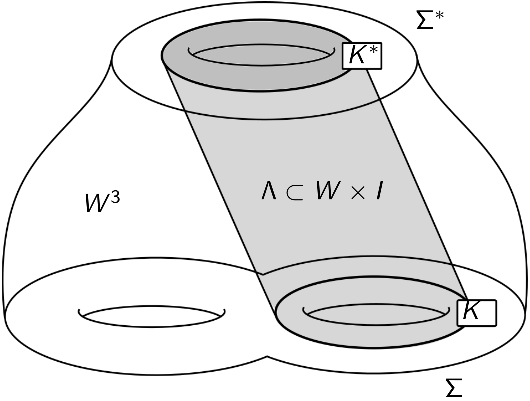
Knots $K$, $K^*$ in $S^3$ are **concordant** if:




**Slice knot/link**:=Concordant to unknot/unlink

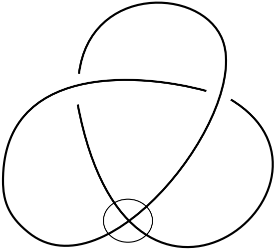
# Virtual concordance (Turaev '08)

Knots $K \subset \Sigma \times I$, $K^* \subset \Sigma^* \times I$ are **virtually concordant** if:

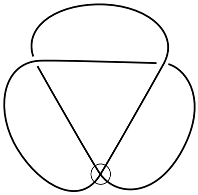



**Virtually slice**:= Concordant to the unknot in $S^2 \times I$

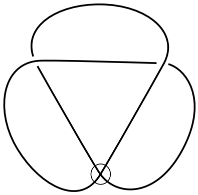
# Virtual knot diagrams

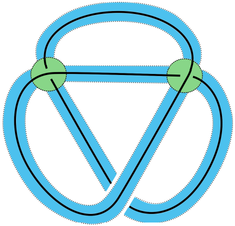


# Virtual knots $\rightarrow$ knots in $\Sigma \times I$.

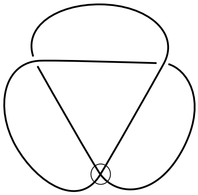


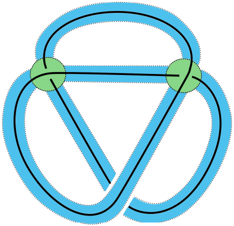
# Virtual knots $\rightarrow$ knots in $\Sigma \times I$.

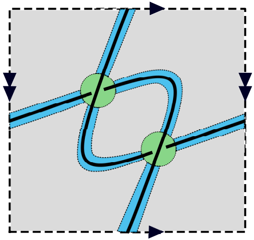




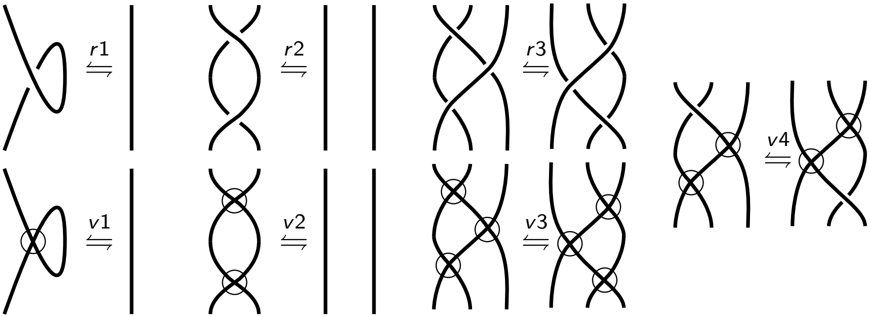
# Virtual knots $\to$ knots in $\Sigma \times I$.



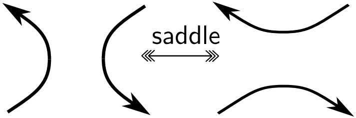


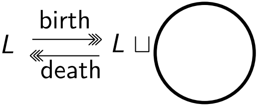


# Extended Reidemeister moves



# Virtual knot concordance (Kauffman '14)



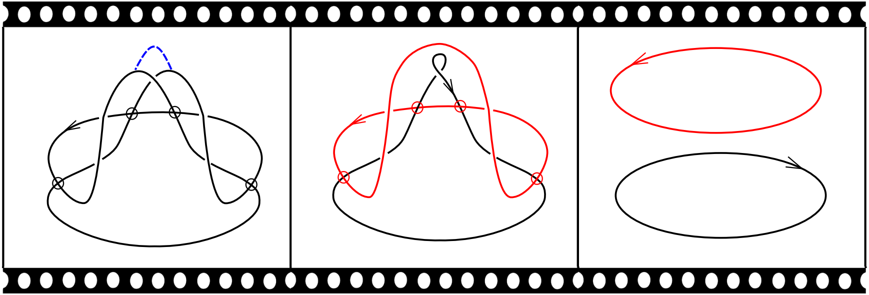




$$\textbf{Virtual knot concordance} := \frac{\text{virtual knot diagrams}}{\left(\begin{array}{c}\text{v-moves+births+deaths+saddles s.t.}\\ \#\text{births} - \#\text{saddles} + \#\text{deaths} = 0\end{array}\right)}$$

**Theorem (Carter-Kamada-Saito '00)**

Two knots in thickened surfaces are virtually concordant if and only if they represent concordant virtual knots.

# Example (5.1216)



# Some facts about virtual concordance

**Theorem (Boden-Nagel '17)**

*Two classical knots in $S^3$ are concordant if and only if they are virtually concordant.*

# Some facts about virtual concordance

**Theorem (Boden-Nagel '17)**

*Two classical knots in $S^3$ are concordant if and only if they are virtually concordant.*

**Theorem (C, https://arxiv.org/abs/1904.05288)**

*Every virtual knot $\upsilon$ is concordant to:*

- *a prime satellite virtual knot,*
- *a prime hyperbolic virtual knot, and*
- *if $\upsilon$ is almost classical (AC) knot, a prime satellite AC knot and a prime hyperbolic AC knot having the same Alexander polynomial.*

# Slice obstructions

## 2008-2019

- (Turaev) graded genus $\vartheta$.
- (Dye-Kaestner-Kauffman) Rasmussen invariant.
- (C-Kaestner) Henrich-Turaev polynomial.
- (Rushworth) a $2^{\text{nd}}$ Rasmussen invariant.
- (Boden-C-Gaudreau 1, Rushworth) odd writhe.
- (Boden-C-Gaudreau 1, Kauffman) writhe polynomial.
- (Boden-C-Gaudreau 2) directed Tristam-Levine signature fncs.
- (Boden-C) generalized Alexander polynomial $\Delta^0$.

# Slice status of v-knots in Green's table

| Crossing number | Virtual knots | $\vartheta = 0$ sieve | $\vartheta = 0$ & $\Delta^0 = 0$ | slice knots | status unknown |
|---|---|---|---|---|---|
| 2 | 1 | 0 | 0 | 0 | 0 |
| 3 | 7 | 1 | 0 | 0 | 0 |
| 4 | 108 | 15 | 14 | 13 | 0 |
| 5 | 2448 | 59 | 51 | 45 | 2 |
| 6 | 90235 | 1476 | 1294 | 1241 | 36 |

- (BCG1,BCG2,BC) Summary of calculations above.
- (White) Calculations of $\Delta^0$.
- (Rushworth, Karimi) Rasmussen invariant calculations.

| | | | |
|---|---|---|---|
| 5.1216 | 5.1963 | 6.5588 | 6.5958 |
| 6.6589 | 6.7070 | 6.7388 | 6.8451 |
| 6.14778 | 6.14781 | 6.15200 | 6.15952 |
| 6.31455 | 6.33334 | 6.37879 | 6.38158 |
| 6.38183 | 6.43763 | 6.46936 | 6.46937 |
| 6.47024 | 6.47172 | 6.47512 | 6.49338 |
| 6.52373 | 6.62002 | 6.69085 | 6.70767 |
| 6.71306 | 6.71848 | 6.72353 | 6.72431 |
| 6.76251 | 6.76488 | 6.77331 | 6.77735 |
| 6.86951 | 6.89218 | | |

# Status unknown

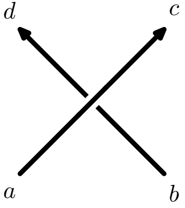
The v-knots in red are slice. (C '20)

| | | | |
|---|---|---|---|
| 5.1216 | 5.1963 | 6.5588 | 6.5958 |
| 6.6589 | 6.7070 | 6.7388 | 6.8451 |
| 6.14778 | 6.14781 | 6.15200 | 6.15952 |
| 6.31455 | 6.33334 | 6.37879 | 6.38158 |
| 6.38183 | 6.43763 | 6.46936 | 6.46937 |
| 6.47024 | 6.47172 | 6.47512 | 6.49338 |
| 6.52373 | 6.62002 | 6.69085 | 6.70767 |
| 6.71306 | 6.71848 | 6.72353 | 6.72431 |
| 6.76251 | 6.76488 | 6.77331 | 6.77735 |
| 6.86951 | 6.89218 | | |

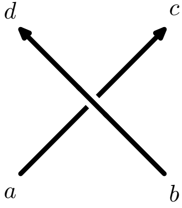
# Extended $\bar{\mu}$-invariants

# Extended group

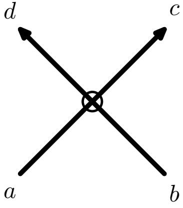
Many equivalent versions in the literature. We will use the following:
(Boden-Gaudreau-Harper-Nicas-White '17)




$c = vav^{-1}$
$d = a^{-1}v^{-1}bva$




$c = bvav^{-1}b^{-1}$
$d = v^{-1}bv$




$c = a$
$d = b$

$$\widetilde{G}(L) \quad := \quad \langle a_1, \dots, a_{2n}, v | r_1, \dots, r_{2n} \rangle$$

This is an extension of the group of a virtual link $L$; just set $v = 1$.

$$G(L) \quad = \quad \langle a_1, \dots, a_{2n}, v | r_1, \dots, r_{2n}, v = 1 \rangle$$

# Extended Chen-Milnor Theorem

**Theorem (C '20)**

*$L$ an $m$-component virtual link. Let $F = F(m+1)$ be the free group on $a_1, \ldots, a_m, v$. The nilpotent quotients of $\widetilde{G} = \widetilde{G}(L)$ are given by:*

$$\widetilde{G}/\widetilde{G}_q \cong \langle a_1, \ldots, a_m, v | [a_1, \widetilde{\lambda}_1^{(q)}], \ldots, [a_m, \widetilde{\lambda}_m^{(q)}], F_q \rangle.$$

# Extended Chen-Milnor Theorem

**Theorem (C '20)**

*$L$ an $m$-component virtual link. Let $F = F(m+1)$ be the free group on $a_1, \ldots, a_m, v$. The nilpotent quotients of $\widetilde{G} = \widetilde{G}(L)$ are given by:*

$$\widetilde{G}/\widetilde{G}_q \cong \langle a_1, \ldots, a_m, v | [a_1, \widetilde{\lambda}_1^{(q)}], \ldots, [a_m, \widetilde{\lambda}_m^{(q)}], F_q \rangle.$$

**Corollary (C '20)**

*If $K$, is a virtual knot, $F = \langle a, v \rangle$, this gives:*

$$\widetilde{G}/\widetilde{G}_q \cong \langle a, v | [a, \widetilde{\lambda}^{(q)}], F_q \rangle.$$

Note: the nilpotent quotients of $G(K)$ are free, but the nilpotent quotients of $\widetilde{G}(K)$ are generally not free.

# Extended $\bar{\mu}$-invariants

$K$ a virtual knot diagram, $J$ be a sequence $\{1, 2\}$, $q > |J|$.

$$\overline{ж}_J(K) \equiv \epsilon_J(\widetilde{\lambda}^{(q)}) \pmod{\Delta_{J|1}}$$

The family of these residue classes are called the $\overline{ж}$-*invariants*.

# Extended $\bar{\mu}$-invariants

$K$ a virtual knot diagram, $J$ be a sequence $\{1,2\}$, $q > |J|$.

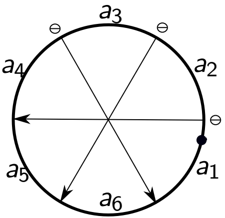
$$\overline{ж}_J(K) \equiv \epsilon_J(\widetilde{\lambda}^{(q)}) \pmod{\Delta_{J|1}}$$

The family of these residue classes are called the $\overline{ж}$-*invariants*.

**Theorem (C '20)**

*The $\overline{ж}$-invariants are concordance invariants of virtual knots.*

# Example (3.5)




3.5

$$
\begin{aligned}
\widetilde{G} = \langle v, a_1, a_2, a_3, a_4, a_5, a_6 | a_2 &= \bar{v} a_1 v, \\
a_3 &= \bar{v} a_2 v, \\
a_4 &= \bar{v} a_3 v, \\
a_5 &= a_1 v a_4 \bar{v} \bar{a}_1, \\
a_6 &= a_2 v a_5 \bar{v} \bar{a}_2, \\
a_1 &= a_3 v a_6 \bar{v} \bar{a}_3 \rangle.
\end{aligned}
$$

# Example (3.5)

## Calculating first non-vanishing $\overline{ж}$-invariants

**(1)** Compute $\widetilde{\lambda}^{(q)}$:

$$\widetilde{\lambda}^{(4)} = v^2\bar{a}\bar{v}^2\bar{a}\bar{v}^2\bar{a}v^2a^3.$$

**(2)** Write $\widetilde{\lambda}^{(q)}$ as a product of commutators.

$$\widetilde{\lambda}^{(4)} \equiv [[v,a],a]^4[[v,a],v]^4 \quad \text{mod } F_4.$$

(e.g. via Hall's Basis Theorem)

**(3)** Use properties of $\epsilon_J$ to calculate $\epsilon_J(\widetilde{\lambda}^{(q)})$ recursively.

# Example (3.5)

| $J$ | $\overline{\text{ж}}_J$ |
|---|---|
| 111 | 0 |
| 112 | 4 |
| 121 | -8 |
| 211 | 4 |
| 122 | -4 |
| 212 | 8 |
| 221 | -4 |
| 222 | 0 |

# Properties

**Definition (Almost classical knot)**

A virtual knot is said to be almost classical if it admits a homologically trivial representative in some $\Sigma \times I$ (i.e. bounds a Seifert surface).

**Theorem (C '20)**

*If $K$ is concordant to an almost classical knot, then all $\overline{ж}$-invariants of $K$ are vanishing.*

**Theorem (C '20)**

*Let $K$ be a virtual knot. If $\bar{\varkappa}_J(K) = 0$ for all sequences $J$, then the generalized Alexander polynomial of $K$ is trivial.*

# Properties

**Theorem (C '20)**

*Let $K$ be a virtual knot. If $\bar{\varkappa}_J(K) = 0$ for all sequences $J$, then the generalized Alexander polynomial of $K$ is trivial.*

**Corollary (C '20)**

*Let $K$ be a virtual knot. If $\bar{\varkappa}_J(K) = 0$ for all sequences $J$, then the odd writhe, Henrich-Turaev polynomial, and affine index (or writhe) polynomial are all trivial.*

# Revisiting the unknown

| $K$ | Gauss code | length of $\widetilde{\lambda}^{(6)}$ | $\widetilde{\lambda}^{(6)}$ mod $F_6$ |
|---|---|---|---|
| 6.6589 | O1-O2-O3-O4+U3-O5+U4+O6+U2-U1-U5+U6+ | 22 | $g_{10}\bar{g}_{11}g_{12}\bar{g}_{13}$ |
| 6.7070 | O1+O2-O3-U2-O4-U3-U5+U4-O6+U1+O5+U6+ | 792 | $\bar{g}_{10}g_{11}$ |
| 6.15200 | O1+O2+O3-O4+U3-O5-O6-U5-U1+U6-U4+U2+ | 20 | $\bar{g}_{10}g_{11}g_{12}\bar{g}_{13}$ |
| 6.15952 | O1-O2-U1-O3-O4+U3-U5+O6+O5+U6+U2-U4+ | 1120 | $g_{10}\bar{g}_{11}\bar{g}_{12}g_{13}$ |
| 6.43763 | O1+U2+O3+O4-O2+U4-O5-U6-U1+U5-O6-U3+ | 586 | $\bar{g}_{10}g_{11}$ |
| 6.47172 | O1+O2-O3-O4-U3-O5+U2-U6+U1+U4-O6+U5+ | 634 | $g_{10}^2\bar{g}_{11}^2$ |
| 6.47512 | O1+O2+O3-O4+U3-O5-U2+U6-U1+U5-O6-U4+ | 934 | $\bar{g}_{10}^2 g_{11}^2$ |
| 6.71848 | O1-O2+O3-U1-O4-U2+O5+U6+U4-U5+O6+U3- | 586 | $g_{10}\bar{g}_{11}$ |
| 6.72431 | O1-O2+O3-U1-O4-U3-O5+U2+O6+U5+U4-U6+ | 14 | $g_{10}\bar{g}_{11}$ |
| 6.76251 | O1-O2+O3-U1-O4-U5+O6+U2+O5+U6+U4-U3- | 498 | $g_{10}\bar{g}_{11}$ |
| 6.89218 | O1-O2+U3+O4+U2+O3+U5-O6-U1-O5-U6-U4+ | 2278 | $\bar{g}_{10}g_{11}$ |

# Revisiting the unknown

The commutator basis used by *ANU NQ* is:

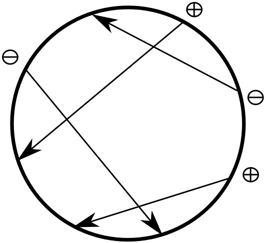
$$
\begin{aligned}
g_1 &= a & g_8 &= [v, a, v, v] \\
g_2 &= v & g_9 &= [v, a, a, a, a] \\
g_3 &= [v, a] & g_{10} &= [v, a, a, a, v] \\
g_4 &= [v, a, a] & g_{11} &= [v, a, v, a, a] \\
g_5 &= [v, a, v] & g_{12} &= [v, a, v, a, v] \\
g_6 &= [v, a, a, a] & g_{13} &= [v, a, v, v, a] \\
g_7 &= [v, a, v, a] & g_{14} &= [v, a, v, v, v]
\end{aligned}
$$

# Revisiting the unknown

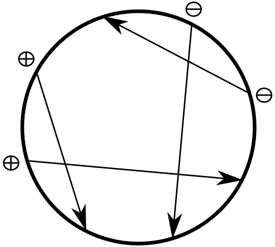
| $J$ | $\bar{Ж}_J(K)$ 6.6589 | 6.7070 | 6.15200 | 6.15952 | 6.43763 | 6.47172 | 6.47512 | 6.71848 | 6.72431 | 6.76251 | 6.89218 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 21111 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 21211 | 1 | -1 | -1 | 1 | -1 | 2 | -2 | 1 | 1 | 1 | -1 |
| 22111 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22121 | 1 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22211 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22221 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Beyond the first non-vanishing degree

These two knots have the same GAP, graded genus, slice genus, and $\bar{ж}$-invariants up to degree 3.



4.19

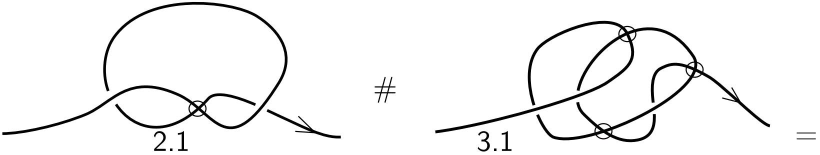


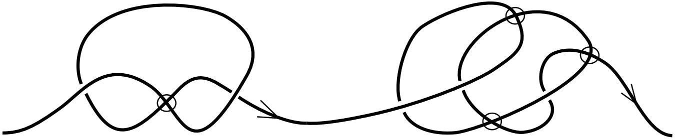
4.32

$$\bar{ж}_{2221}(K) \equiv 1 \pmod 2 \quad \bar{ж}_{2211}(K) \equiv 0 \pmod 2 \quad \bar{ж}_{2111}(K) \equiv 1 \pmod 2$$
$$\bar{ж}_{2221}(K^*) \equiv 1 \pmod 2 \quad \bar{ж}_{2211}(K^*) \equiv 1 \pmod 2 \quad \bar{ж}_{2111}(K^*) \equiv 0 \pmod 2$$

# Virtual knot concordance group

# Concatenation



# Definition & background

**Virtual concordance group**

$\mathcal{VC} :=$ (concordance classes of long virtual knots, $\#, 1 =$ long unknot)

# Definition & background

**Virtual concordance group**

$\mathcal{VC} :=$ (concordance classes of long virtual knots, $\#, 1 =$ long unknot)

**Theorem (Boden-Nagel '17)**

*The classical knot concordance group $\mathcal{C}$ embeds into $\mathcal{VC}$.*

# Definition & background

**Virtual concordance group**

$\mathcal{VC} :=$ (concordance classes of long virtual knots, $\#, 1 =$ long unknot)

**Theorem (Boden-Nagel '17)**

*The classical knot concordance group $\mathcal{C}$ embeds into $\mathcal{VC}$.*

**Turaev '08**

Question: "Is it abelian?"

# Definition & background

**Virtual concordance group**

$\mathcal{VC} :=$ (concordance classes of long virtual knots, $\#, 1 =$ long unknot)

**Theorem (Boden-Nagel '17)**

*The classical knot concordance group $\mathcal{C}$ embeds into $\mathcal{VC}$.*

**Turaev '08**

Question: "Is it abelian?"

**Theorem (Manturov '08)**

*Equivalence classes of long virtual knots are a non-commutative monoid.*

# Extended Artin representation

**Theorem (C)**

*Let $\vec{K}$ be a long virtual knot, $\widetilde{G} = \widetilde{G}(\vec{K})$, and $F$ the free group on $a, v$. For all $q \geq 2$, there is a isomorphisms on nilpotent quotients:*

$$\widetilde{G}/\widetilde{G}_q \xrightarrow{\cong} F/F_q.$$

# Extended Artin representation

**Theorem (C)**

*Let $\vec{K}$ be a long virtual knot, $\widetilde{G} = \widetilde{G}(\vec{K})$, and $F$ the free group on $a, v$. For all $q \geq 2$, there is a isomorphisms on nilpotent quotients:*

$$\widetilde{G}/\widetilde{G}_q \xrightarrow{\cong} F/F_q.$$

Following Habegger-Lin '98, we define:

$$A^{(q)}_{\text{ж}} : \mathcal{VC} \to \mathrm{Aut}(F/F_{q+1})$$
$$A^{(q)}_{\text{ж}}(\vec{K})(v) = v$$
$$A^{(q)}_{\text{ж}}(\vec{K})(a) = \widetilde{\lambda}^{(q)} a (\widetilde{\lambda}^{(q)})^{-1}$$

# Results on $\mathcal{VC}$

**Theorem (C '20)**

$A^{(q)}_{\text{ж}}$ *is a concordance invariant of long virtual knots.*

# Results on $\mathcal{VC}$

**Theorem (C '20)**

*$A^{(q)}_{ж}$ is a concordance invariant of long virtual knots.*

**Theorem (C '20)**

*$\mathcal{VC}$ is not abelian.*

**Proof.**

$A^{(8)}_{ж}(2.1\#3.1) \neq A^{(8)}_{ж}(3.1\#2.1)$ $\square$

# Results on $\mathcal{VC}$

**Theorem (C '20)**

$A^{(q)}_{\text{ж}}$ *is a concordance invariant of long virtual knots.*

**Theorem (C '20)**

$\mathcal{VC}$ *is not abelian.*

**Proof.**

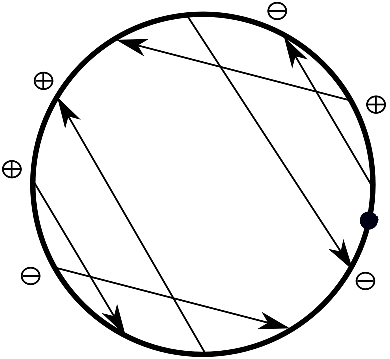
$A^{(8)}_{\text{ж}}(2.1\#3.1) \neq A^{(8)}_{\text{ж}}(3.1\#2.1)$ □

**Corollary (C '20)**

*There exist non-concordant long virtual knots with concordant closure.*

# Twelfth knot

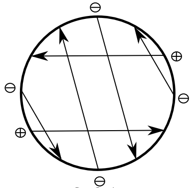
The extended Artin representation obstructs this v-knot from being slice.



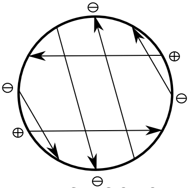

6.8451

# Current unknown list

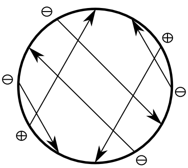
This leaves the following:



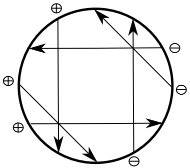
6.31445



6.52373



6.62002



6.86951

# Why is it true?

# Extended $\bar{\mu}$-invariants

Ж =Bar-Natan's Ж map, Tube = Satoh's map.

## Outline of proof

- First, a geometric realization:

$$\text{v-knots} \xrightarrow{\text{Ж}} \text{w-links} \xrightarrow{\text{Tube}} \text{Ribbon torus links in } S^4$$

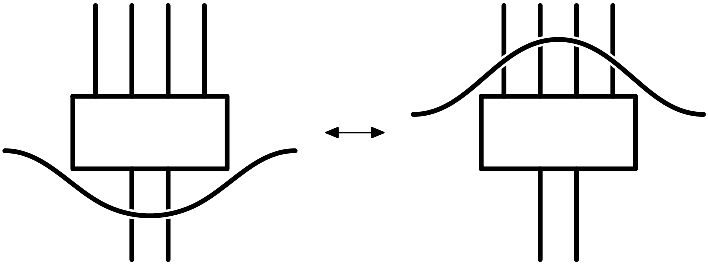
- These induce isomorphisms:

$$\widetilde{G}(K) \cong G(\text{Ж}(K)) \cong \pi_1(S^4 \smallsetminus \text{Tube}(\text{Ж}(K)), *)$$

- (Boden-C, '19) Ж is functorial under concordance.
- (Boden-C, '19) Tube is functorial under concordance.
- (C, '20) Generalize $\bar{\mu}$-concordance invariants to welded links.
- The $\bar{\text{ж}}$-invariants are: Milnor+Tube+Ж.

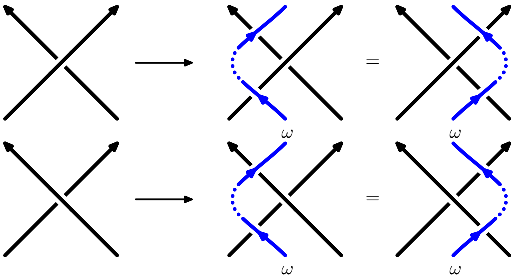
# Welded knots and “overcrossings commute”

$$\textbf{Welded knots} := \frac{\text{virtual knots}}{\text{“overcrossings commute”.}}$$



# The Bar-Natan Ж ("Zh") map

Add a new component $\omega$ to make a semi-welded link.

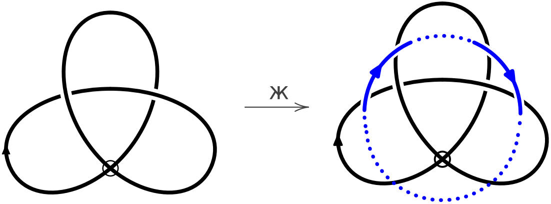



**How to form the extra component $\omega$**

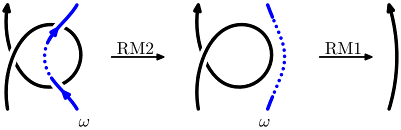
Glue together the arcs ends arbitrarily, new crossings are virtual.

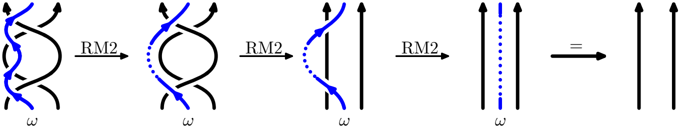
This is well-defined since "overcrossings commute" in $\omega$.

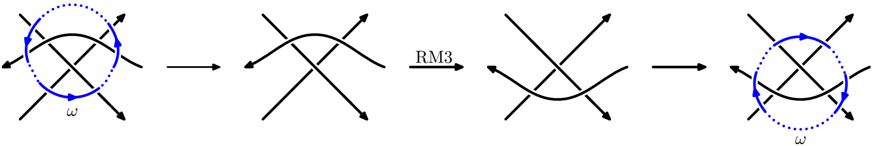
# Example



# Invariance under Reidemeister moves



# Lemmas for Ж

**Lemma (Boden-C '19)**

Ж *maps concordant v-knots to concordant semi-welded links.*

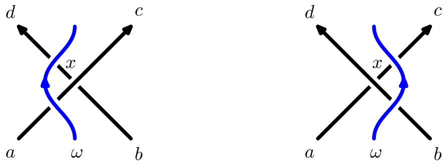
# Lemmas for Ж

**Lemma (Boden-C '19)**

*Ж maps concordant v-knots to concordant semi-welded links.*

**Lemma (Boden-C '19)**

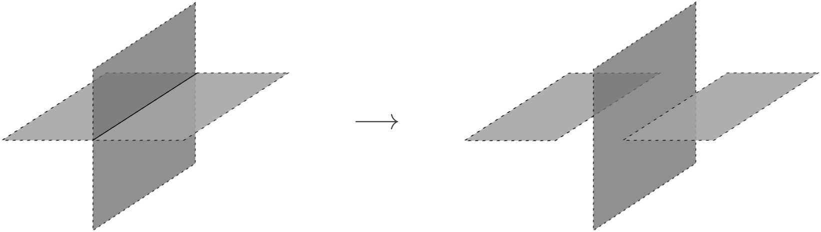
*For all virtual knots $K$, $\widetilde{G}(K) \cong G(Ж(K))$.*

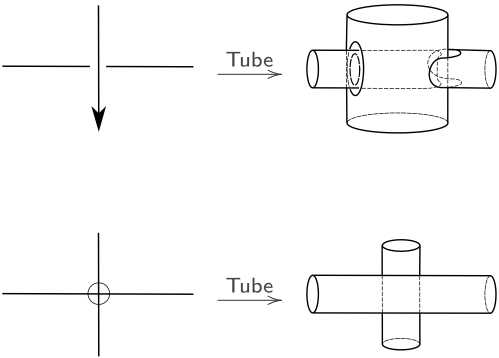



$c = \omega a \omega^{-1}$
$x = c^{-1} b c$
$d = \omega^{-1} x \omega = a^{-1} \omega^{-1} b \omega a$

$d = \omega^{-1} b \omega$
$x = d a d^{-1}$
$c = \omega x \omega^{-1} = b \omega a \omega^{-1} b^{-1}$

# Satoh's Tube map I: Broken surface diagrams




Tube
Tube
Tube

# Properties of Tube

**Theorem (Satoh '00)**

*$L$, $L^*$ equivalent w-links $\implies$ $\mathit{Tube}(L)$, $\mathit{Tube}(L^*) \subset S^4$ are isotopic.*

# Properties of Tube

**Theorem (Satoh ‘00)**

*$L$, $L^*$ equivalent w-links* $\implies$ $\mathit{Tube}(L)$, $\mathit{Tube}(L^*) \subset S^4$ *are isotopic.*

**Theorem (Satoh ‘00)**

*For any welded link $L$,*

$$G(L) \cong \pi_1(S^4 \setminus \mathit{Tube}(L), *).$$

# Concordance invariance of Tube

**Definition (Torus Link Concordance)**

A *concordance* of ribbon torus links $T_0, T_1 \subset S^4$ is a smooth proper embedding $W$ in $S^4 \times I$ with finitely many components, each diffeomorphic to $(S^1 \times S^1) \times I$, and $W \cap (S^4 \times \{i\}) = T_i$ for $i = 0, 1$.

**Theorem (Boden-C '19)**

*$L, L^*$ concordant w-links $\implies$ Tube($L$),Tube($L^*$) concordant.*

**Lemma (C '20)**

*If $L, L^*$ welded-concordant virtual links, $\Lambda$ a concordance between $Tube(L), Tube(L^*)$, then:*

$$Q_q(\pi_1(S^4 \smallsetminus T)) \cong Q_q(\pi_1(S^4 \times I \smallsetminus \Lambda)) \cong Q_q(\pi_1(S^4 \smallsetminus T^*))$$

$$Q_q(G(L)) \cong Q_q(\pi_1(S^4 \times I \smallsetminus \Lambda)) \cong Q_q(G(L^*))$$

*and the isomorphisms preserve longitude words. Here $Q_q(A)$ denotes the q-th nilpotent quotient of $A$.*

**Lemma (C '20)**

*If $L, L^*$ welded-concordant virtual links, $\Lambda$ a concordance between Tube$(L)$, Tube$(L^*)$, then:*

$$Q_q(\pi_1(S^4 \setminus T)) \cong Q_q(\pi_1(S^4 \times I \setminus \Lambda)) \cong Q_q(\pi_1(S^4 \setminus T^*))$$

$$Q_q(G(L)) \cong Q_q(\pi_1(S^4 \times I \setminus \Lambda)) \cong Q_q(G(L^*))$$

*and the isomorphisms preserve longitude words. Here $Q_q(A)$ denotes the q-th nilpotent quotient of A.*

**Proof.**

Apply Stalling's theorem. □

# $\bar{\mu}$-invariants for welded links

**Theorem (C '20)**

*If $L, L^*$ are welded-concordant virtual links (or concordant ribbon torus links in $S^4$), then for all sequences $J$ with $|J| \geq 2$:*

$$\bar{\mu}_J(L) \equiv \bar{\mu}_J(L^*) \pmod{\Delta_J}.$$

# All together now

**Recall that each map is functorial under concordance**

$$\text{v-knots} \xrightarrow{\text{Ж}} \text{w-links} \xrightarrow{\text{Tube}} \text{Ribbon torus links in } S^4$$

# All together now

**Recall that each map is functorial under concordance**

$$\text{v-knots} \xrightarrow{\text{ж}} \text{w-links} \xrightarrow{\text{Tube}} \text{Ribbon torus links in } S^4$$

$$\overline{\text{ж}} = \bar{\mu}(\text{Ж})$$

# All together now

**Recall that each map is functorial under concordance**

$$\text{v-knots} \xrightarrow{\text{ж}} \text{w-links} \xrightarrow{\text{Tube}} \text{Ribbon torus links in } S^4$$

$$\overline{\text{ж}} = \bar{\mu}(\text{Ж})$$

$\therefore$ $\bar{\text{ж}}$-invariants are concordance invariants of v-knots.

Thank you!