SYLLABUS: MATH 415 REAL ANALYSIS, SPRING 2014

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Office Hours:	W-11:30-12:30,Th-10-11,F-8:30-9:30.

Textbook: ..... *Real Analysis: A First Course*, by Russel A. Gordon

Course Requirements: Your grade in the course will be based on your performance on the following assignments.

(1) *Reading Assignment:* You are expected to read the section of the text indicated on the *Day-To-Day Assignment Sheet* before coming to class. A brief 30 minute preview will be sufficient.
(2) *Homework:* You will have one HW assignment every two weeks which will be submitted digitally using the UHAS (see FAQ).
(3) *Definition Quizzes:* You will have five definition quizzes. I will give you five definitions recently discussed in class. Of the five, I will choose three randomly and you will have to write them out verbatim.
(4) *Chapter Papers:* There are three Chapter Papers, each of length between four and five pages. A Chapter Paper is, roughly, a summary of the main ideas in a chapter from your text which is illustrated with two (corrected) HW problems. This satisfies the Writing Intensive Requirement.
(5) *Tests/Cumulative Final:* The tests an final exam will have definitions, computations, True/False questions, and proofs. For example, you might be given four theorems on a test. You may choose any two of them to prove.

Grading: The break down is as follows:

- Homework 15%
- Def Quiz 5 ..... 5%
- Chapter Papers 20%
- 2 Tests 40%
- Cumulative Final Exam 20%

Important Dates: Due dates for assignments are on the Day-To-Day Assignment Sheet. Other important dates are:

- Test 1 ..... February 21, 2014
- Undergraduate Midterm Grades ..... March 11, 2014
- Withdraw w/ "W" Deadline ..... March 31, 2014
- Test 2 ..... April 11, 2014
- Cumulative Final Exam ..... TBA

Final Course Grades: Let X be the numerical grade determined by your individual percentages as above. At the end of the semester, X is associated to a letter grade α. The scheme is:

- $X \in [.9, \infty)$ ..... $\alpha \in \{A-, A\}$.
- $X \in [.8, .9)$ ..... $\alpha \in \{B-, B, B+\}$.
- $X \in [.7, .8)$ ..... $\alpha \in \{C-, C, C+\}$.
- $X \in [.6, .7)$ ..... $\alpha \in \{D-, D, D+\}$.
- $X \in (-\infty, .6)$ ..... $\alpha = F$.

Extra Credit: You may attend any of the Undergraduate Mathematics Colloquiums during the Spring 2014 semester for extra credit. Type up a clear and short ($\approx$ 1 page) summary of the speaker's talk. Put your file in the correct eCampus dropbox.

NO OTHER EXTRA CREDIT IS OFFERED.

Attendance: You are expected to attend every class. You will not be allowed to make-up assignments you have missed without prior excuse or proof of serious circumstances beyond your control (e.g. sickness, death in the family...). If you cannot make it to class, please call or e-mail me no later than the day of class that you have missed. If you have not registered, please see me immediately.

Decorum: The basic tenet of your attendance in this class is: Do no harm to the learning of other students. This includes talking with students while I am lecturing, interrupting me while I am lecturing, distracting computer use (e-mail, cellphone, Internet,...), snide comments, working on other coursework, cursing, harassment, socializing during group activities, etc. If you violate this principle, you will be warned once. On the second infraction you will be removed from the classroom.

Special Accommodations: Please inform me as soon as possible. We are happy to make reasonable accommodations. Please see the "University Guide to Services and Accommodations for Students with Disabilities" for more information.

Make-Up Exams/Late Work: You may not retake a test for any unexcused absence. If a missed test results from an excused absence, then you must make an appointment with me to retake the test. Arrangements must be made prior to the start of the next class meeting. No make-up exams will be given more than a week after the scheduled exam. Make-up exams will be significantly longer than the scheduled exam (at least 25%), as you have had more time to prepare than the rest of the class. You may turn in exactly one late HW assignment during the semester. If you just turn in one problem late, then the ENTIRE ASSIGNMENT IS LATE. See the UHAS FAQ.

Academic Honesty: I encourage you to work with other students on homework and projects. However, you are expected to turn in your own work in your own words. You cannot copy (either physically or electronically) another student's work and turn it in as your own. Note that this is especially important because you are all turning in all HW assignments digitally. If you "copy", in any sense of the word, another students assignment exactly, you and the the person who copied the assignment will both receive a zero on the entire assignment. Please see the Student Code of Conduct for a list of all prohibited practices. Some additional rules:

(1) *Group Work*: Group work is prohibited for Chapter Papers. These are to be assignments which reflect your individual understanding of the topic.
(2) *Definition Quizzes*: No electronic devices of any kind are permitted to be used during a Def Quiz. Medical devices are an obvious exception to this rule.
(3) *Tests:* Scientific and graphing calculators are permitted, but will generally not be necessary. Otherwise, no electronic devices of any kind are permitted (excepting medical devices).

Real Analysis Study Suggestions: Here are some tips for learning real analysis.

(1) Do your HW conscientiously. It is much easier to do a couple of problems a night than to do a bunch of problems the night before the assignment is due.
(2) Discipline and fearlessness are much better traits than brilliance.
(3) Office hours are a part of the course. You're paying for them, so please take advantage of my help.
(4) Read the text before you come to class. You don't have to understand all of it at first. Read it like a novel.
(5) You will need to study each new idea several times before you can begin to understand it.
(6) It is helpful if you write down all of the definitions and theorems on a separate sheet of paper as you go through the class. Studying for tests and quizzes will take less time.

CHAPTER PAPER FAQ

You have three *Chapter Papers* to submit this semester: CH1 Paper, CH2 Paper, CH3/4 Paper. The due dates on the *Day-To-Day Assignment Sheet* which may be found on eCampus.

1.1. **How long is a Chapter Paper?** Word counts for each of the sections are given below. It will probably be somewhere between 4 and 5 pages. No funny business with the margins or font size.

1.2. **Does the Chapter Paper need to be typed?** Yes. Use either Microsoft or LaTeX.

1.3. **What is the content of a Chapter Paper?** Each Chapter Paper must contain all of the following. Please read this very carefully each time before submitting a Chapter Paper.

(1) An Introduction ($\geq$ 125 words).

(2) A summary of the important ideas in the chapter(s). Ask yourself: what is this chapter about? What were the key proof techniques developed in this chapter(s)? What important results from calculus were established in this chapter? ($\geq$ 500 words)

(3) After the summary, take two HW problems from the chapter. If you missed any problems from this chapter(s), you must include problems that you missed. You cannot submit a correct problem unless you missed < 2 problems from the chapter(s). When you have picked your problems, make any corrections necessary. IMPORTANT: Include a statement of each of the HW problems. Do not simply write down the problem number.

(4) For each of the chosen problems, come up with a simple example which illustrates the problem. For example, suppose the problem is to prove that if a function f is differentiable at a point c point and has a local extrema at c, then $f'(c) = 0$. An example is $f(x) = x^2$ with $c = 0$. Be sure to spell out all of the details as to why the given example is an illustration of the result.

(5) A conclusion. In your conclusion, discuss how each of your chosen problems highlights at least one main idea from the chapter(s). If it highlights more than one idea, then please so and explain why. ($\approx$ 200 words)

Remark 1.1. Occasionally, I will tell you that certain problems cannot be included in a Chapter Paper. This may be because they are too computational or because they simply not proof-type problems.

1.4. **How do I submit my Chapter Paper?** Put it in the eCampus folder with the appropriate title. If you put it in the wrong folder, you will not receive any credit.

1.5. **Can't I just e-mail you my Chapter Paper?** No.

1.6. **What is your policy on late work?** You may submit a late Chapter Paper, but it will count as your one freebie for the semester.

WARNING: Once the Dropbox closes, it will not be reopened for any reason. All late Chapter Papers must be submitted to the LATE dropbox.

1.7. **Can we do our papers in groups?** No. A Chapter Paper is designed to be an individual reflection on the material. You should avoid discussing the content of your Chapter Paper with other students in the class. If you turn in a paper which is partly identical to another paper in the class, then both papers will receive a zero.

1.8. **By what rubric will I be graded?** Each Chapter Paper will be graded according to the following rubric.

- (2 pts.) Introduction.
- (5 pts.) Summary of main ideas in chapter(s).
- (5 pts.) HW Problem 1 w/corrected proof.
- (2 pts.) Example illustrating HW problem 1.
- (5 pts.) HW Problem 2 w/corrected proof.
- (2 pts.) Example illustrating HW problem 2.
- (4 pts.) Conclusion discussing how the two problems you picked are related to the main ideas of the chapter(s).

1.9. **How will I be given feedback on my assignment?** You will receive feedback directly on eCampus as soon as I finish grading your assignment. Note that if somebody else has a graded assignment and you do not, it probably means that I have not finished grading all of the papers yet. Be patient! If it seems like a long time has passed and you don't have a grade, please let me know.

1.10. **Rationale for Chapter Papers.** Here are some reasons why I think Chapter Papers are a good idea.

(1) It meets Monmouth University requirements for Writing Intensive courses.

(2) It encourages you to think in terms of the big picture. It is easy to lose sight of what you are doing when you are trying to get your HW assignment completed. This gives you a chance to reflect on *why* you needed to do certain HW problems. Eventually, you will forget all of the details and remember only the big picture anyway.

(3) As with the UHAS, this solves the problem of distribution of papers.

(4) It will make it easier to study for tests and the final exam. You will have the main ideas already figured out, so then you just need to practice *using* the main ideas.

LECTURE 1: REAL NUMBERS, THEIR PROPERTIES

These are notes to go along with the book "Real Analysis: A First Course", by Russel A. Gordon.

The following notation will be used throughout the course:

$$\begin{aligned}\mathbb{N} &= \{1, 2, 3, \ldots\} \\ \mathbb{Z} &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\ \mathbb{Q} &= \left\{\frac{a}{b} \middle| a, b \in \mathbb{Z}, b \neq 0\right\}\end{aligned}$$

There are other numbers which should be called "real numbers" but are not rational numbers: $\sqrt{2}, \pi, e$.

Theorem 1. *The number $\sqrt{2}$ is not a rational number.*

Proof. Suppose that $\sqrt{2}$ is a rational number. Then $\sqrt{2} = m/n$ for some $m, n \in \mathbb{Z}$. We can assume without loss of generality that m and n have no common factors and are both positive. Then $m^2 = 2n^2$. Since 2 is prime, $2|m$. Thus, $4|m^2$. Hence, $4k = m^2$ for some $k \in \mathbb{N}$. Then $2k = n^2$ and $2|n$. This contradicts the assumption that m and n have no common factors. Thus, $\sqrt{2}$ is irrational. □

We think that $\sqrt{2}$, π and e should be "real numbers" because they are approximated by rational numbers. From experience, we know that the following lists of numbers get closer to π and $\sqrt{2}$:

$$\begin{aligned}3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \ldots &\to \pi \\ 1, 1.4, 1.41, 1.414, 1.4142, 1.41423, 1.414234, \ldots &\to \sqrt{2}\end{aligned}$$

This is a bad way to define an approximation since, for example, it is very hard to figure out the decimal digits of π. What is a real number, anyway????

As in mentioned in the text, you have probably only thought of real numbers in terms as either decimal expansions or as points ordered on a line. Both are important ways to think about real numbers, but they don't define real numbers.

1.1. **Decimal Expansions.** Every real number has a decimal expansion (we will prove this later). Every rational number has a finite or repeating decimal expansion (this is the division algorithm).

Conversely, every repeating decimal expansion corresponds to a rational number. For example, $1 = 0.9999\ldots$. There is a simple proof of this fact, which provides interesting entertainment at parties. Let $x = .999\ldots$. Then $10x = 9.999\ldots$. Hence, $10x - 9 = x$. Solving for x gives $x = 1$!

You can do this for any kind of repeating decimal. It gives an algorithm for finding the

rational number corresponding to a repeating decimal expansion. Suppose that we have the repeating decimal $x = 0.a_1a_2a_3 \dots a_n a_1 \dots$

(1) Multiply x by 10^n.
(2) Subtract $a_1a_2 \dots a_n$.
(3) This equals x.
(4) Solve for x.

Can you think of an easier way to do this now?

Example 1.1. Find the the rational numbers corresponding to .123412341234 . . ., .11112341234, and 1.00012341234 . . ..

Theorem 2. *A number is rational if and only if it has a finite or repeating decimal expansion.*

Example: The number .101001000100001000001 . . . is irrational (i.e. not rational).

The reason why decimal expansions give a bad definition for the real numbers is that there are some real numbers whose decimal expansions cannot be expressed by nice patterns like the above.

1.2. **The real line.** The other way you have seen is that real numbers are defined by sequence of points on line.

(1) Draw a line.
(2) Draw a point and label it 0. Draw a point to the right of the point marked 0 and label it 1.
(3) Now you an find any integer.
(4) Now you can find any rational number.
(5) The irrationals are "everything else".

This is no good either. Where is π? Where is e? Where is $\pi + e$? What about$\pi \cdot e$?

1.3. **Axiomatic Characterization.** It is generally easier to characterize mathematical objects by the properties that they possess:

(1) The number $\sqrt{2}$ is the positive solution to $x^2 - 2$.
(2) The number π is the ratio of the circumference of any circle to its diameter.
(3) The number e is the unique real number such that:

$$\int_1^e \frac{1}{t}\, dt = 1$$

What properties do the real numbers have? The algebraic structure of the real numbers is that of a *field*.

Definition 1.1 (Field)**.** A *field* is a set $(F, +, \cdot, 1, 0)$ with two binary operations called addition and multiplication and two unary operations (i.e. constants) 1 and 0 satisfying:

(1) For all $a, b \in F$ $a + b$ and $a \cdot b$ are also in F.
(2) (Associativity of $+$) $a + (b + c) = (a + b) + c$, for all $a, b, c \in F$
(3) (Associativity of $\cdot$) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(4) (Commutativity of $+$) $a + b = b + a$ for all $a, b \in F$.
(5) (Commutativity of $\cdot$) $a \cdot b = b \cdot a$ for all $a, b \in F$
(6) (Distributivity) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$

(7) (Additive Identity) $x + 0 = x$ for all $x \in F$.
(8) (Multiplicative Identity) $a \cdot 1 = a$ for all $a \in F$.
(9) (Existence of Additive Inverses) For all $x \in F$, there is a y such that $x + y = 0$
(10) (Existence of Multiplicative Inverses) for all $a \in F$, $a \neq 0$, there is a $b \in F$ such that $a \cdot b = 1$.

Definition 1.2 (Ordered Set)**.** An *ordered set* is a set S together with a relation $<$ which satisfies the following two properties:

(1) For all $a, b \in S$, either $a < b$, $a > b$, $a = b$.
(2) (Transitive Property) For all $a, b, c \in S$, if $a < b$ and $b < c$ then $a < c$.

Definition 1.3 (Ordered Field)**.** An *ordered field* is a field with an ordering satisfying the following compatibility relations with the operations.

(1) If $x > 0$ and $y > 0$, then $x + y > 0$.
(2) If $x > 0$ and $y > 0$, then $x \cdot y > 0$.
(3) $x < y$ if and only if $y - x > 0$.

Example 1.2. Both the real numbers and the rationals are ordered fields. However, we can distinguish the reals and the rationals by looking at roots of equations. The polynomial $x^2 - 2$ has no root in $\mathbb{Q}$ but has two roots in $\mathbb{R}$. The number π is not the root of any polynomial with rational coefficients, but $x - \pi$ has a root in $\mathbb{R}$. What about $\mathbb{C}$ with the lexicographic ordering? It is field with an ordering, but is it an ordered field?

Remark 1.1. We will discuss a property called the *Completeness Axiom* which distinguishes the real numbers from the rational numbers. It allows us to talk about π and $\sqrt{2}$.

1.4. **Some problems for the class, with answers.** The students should be able to do 1, 2, 3, 4, 11, 13, 16.

(#15 from text 1.1) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Proof. Suppose not. Then $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ is also rational. But then, $2\sqrt{6} = 5 + 2\sqrt{6} - 5$ is rational. This means that $\sqrt{6}$ is rational, which we know is false (by problem #11 in the text from 1.1). □

(# 14 from text 1.1) Prove there is no rational number r such that $2^r = 3$.

Proof. Suppose otherwise. Then $2^{p/q} = 3$ for some integers p and q. Hence $2^p = 3^q$. But this cannot be since $2 \not| 3$ unless $p = q = 0$. Such an r is not rational. This is a contradiction. □

(#19 from text 1.1) In the axioms for an ordered field, show that (3) $\implies$ (1).

Proof. Suppose that $x > 0$ and $y > 0$. We have $x + y - x = y > 0$. Then by property (3), $x + y > x$. Since $x > 0$, the transitive property says that $x + y > 0$. This establishes (1). □

(# 20 from text 1.1) Use the properties of an ordered field to prove that if $x < y$ and $z > 0$, then $xz < yz$.

Proof. Suppose $x < y$ and $z > 0$. Then by (3), $0 < y - x$. By (2), $z(y - x) > 0$. This implies that $zy - zx > 0$, by the distributive property. By (3), we see that $zy > zx$. The statement is proved. □

LECTURE 2: ABSOLUTE VALUE, INTERVALS, MISCELLANEOUS

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

2.1. **Absolute Value.** The absolute value function is used in analysis to express the distance between two points. The notion of distance, or the existence of a **metric** is an extremely important property of the real numbers. What does distance mean? What are the abstract properties of a metric?

Definition 2.1. A **metric space** is a set S along with a function $d : S \times S \to \mathbb{R}^+ \cup \{0\}$ such that:

(1) $d(x, y) = 0$ if and only if $x = y$
(2) (Symmetric) $d(x, y) = d(y, x)$ for all $x, y \in S$
(3) (Triangle Inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in S$.

Metric spaces are important in the study of topology and geometry. We won't say anything more about them other than to show that the absolute value function satisfies these abstract properties.

Recall that the absolute value function is defined for $x \in \mathbb{R}$ as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if} x < 0 \end{cases}$$

Theorem 1 (Properties of the absolute value function(as in text))**.**

(1) $-|a| \leq |a| \leq |a|$ *for all* $a \in \mathbb{R}$
(2) $|a \cdot b| = |a||b|$ *for all* $a, b \in \mathbb{R}$
(3) $|-a| = |a|$ *for all* $a \in \mathbb{R}$
(4) Let $c > 0$*. Then* $|a| < c$ *if and only if* $-c < a < c$ *and* $|a| \leq c$ *if and only if* $-c \leq a \leq c$*.*

The proof of this theorem will be left as an exercise.

Theorem 2 (Triangle Inequality)**.** *For all* $x, y \in \mathbb{R}$*,*

$$|x + y| \leq |x| + |y|$$

Proof. Consider $|x|$ and $|y|$. Since $|x| \leq |x|$, we have

$$-|x| \leq x \leq |x|, \ -|y| \leq y \leq |y|$$

Adding these two expressions gives:

$$-|x| - |y| \leq x + y \leq |x| + |y|$$

By the last property above, we get $|x + y| \leq |x| + |y|$. □

Corollary 3 (Reverse Triangle Inequality)**.** *For all* $a, b \in \mathbb{R}$*,*

$$||a| - |b|| \leq |a - b|$$

Proof. This is a good example of the Add-and-Subtract-the-Same-Thing Trick:

$$\begin{aligned} |a| &= |a-b+b| \\ &\leq |a-b|+|b| \end{aligned}$$

Thus, $|a|-|b| \leq |a-b|$. We can do the same trick starting with $|b|$ by adding and subtracting a on the inside of the absolute value. This gives: $|b|-|a| \leq |a-b|$. Putting it all together, we get the desired result: $||a|-|b|| \leq |a-b|$. □

Corollary 4. *Define a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x,y) = |x-y|$. Then $(\mathbb{R}, d)$ is a metric space.*

Proof. It must be shown that the axioms are satisfied.

(1) If $d(x,y) = 0$, then $|x-y| = 0$. Then $x - y = 0$. Thus $x = y$.
(2) $d(x,y) = |x-y| = |-(x-y)| = |y-x| = d(y,x)$
(3) We use the add-and-subtract trick:

$$\begin{aligned} d(x,y) &= |x-y| \\ &= |x-z+z-y| \\ &\leq |x-z|+|z-y| \\ &= d(x,z)+d(z,y) \end{aligned}$$

□

Example: Suppose that $|x-8| < \frac{1}{2}$. How big is $|x+8|$? This is a kind of thing we will do over and over again in this course. It helps to draw a number line.

$$-\frac{1}{2} < x-8 < \frac{1}{2}$$

$$16-\frac{1}{2} < x+8 < 16+\frac{1}{2}$$

Now, $16 - .5 > -(16 + .5)$. Thus, $|x+8| < \frac{33}{2}$.

Theorem 5. *Let $x \in \mathbb{R}$. If $|x| < \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Proof. Suppose otherwise. Then one of x and $-x$ is larger than 0. Hence, $|x| > 0$. By our order properties, $|x|/2 > 0$. Thus:

$$0 < |x| < \frac{|x|}{2}$$

This tells us that $-|x|/2 > 0$, which is a contradiction. We conclude that $x = 0$. □

2.2. **Intervals.**

Definition: An subset $S \subset \mathbb{R}$ is said to be an interval if it contains at least two points and has the property that if $x, y \in S$, and $y < z < x$ for some real number z, then $z \in S$.

I disagree with the text here on the discussion of the symbol ∞. It should be stressed that ∞ *is not a number*. It is just a symbol that represents the absence of a bound. The use of the symbol ∞ takes on many different things in many different contexts. We will talk about one of these contexts when we discuss countable and uncountable sets.

Intervals can be classified using either topology or the Completeness Axiom. The following are obviously intervals; we will establish the converse in the next section.

Theorem 6 (as in text)**.** *An interval has one of the following forms:*

(1) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$*, bounded, open interval*
(2) $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$*, bounded, half-open*
(3) $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$*, bounded, half-open*
(4) $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$*, bounded, closed*
(5) $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$*, unbounded, open.*
(6) $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$*, unbounded, closed*
(7) $(b, \infty) = \{x \in \mathbb{R} : b < x\}$*, unbounded, open*
(8) $[b, \infty) = \{x \in \mathbb{R} : a < x \leq b\}$*, unbounded, closed*
(9) $(-\infty, \infty) = \mathbb{R}$*, unbounded, open, AND closed.*

2.3. **Miscellaneous.**

Theorem 7 (Geometric Sum)**.** *For any real numbers* a, r*,*

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \begin{cases} na & \text{if } r = 1 \\ a \cdot \frac{1-r^n}{1-r} & \text{if } r \neq 1 \end{cases}$$

Proof. The $r = 1$ case is really easy. Assume $r \neq 1$. This is a great trick. Just multiply it out!

$$\begin{aligned} (1-r) \cdot (a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}) &= a + ar + ar^2 + ar^3 + \ldots + ar^n - r(a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}) \\ &= a + (ar - ar) + (ar^2 - ar^2) + \ldots (ar^{n-1} - ar^{n-1}) - ar^n \\ &= a(1 - r^n) \end{aligned}$$

Divide both sides by $(1 - r)$ and you are done. □

Example Find the following sums:

(1) $1 + 2 + 4 + 8 + \ldots + 1024$
(2) $1024 + 3 + 512 + 9 + 256 + 27 + \ldots + 2 + 59049$
(3) $16 - 8 + 4 - 2 + 1 - \frac{1}{2} + \ldots + \frac{1}{1024}$

Also in this section, the text introduces the symbols:

$$\max\{a_1, a_2, \ldots, a_m\}$$
$$\min\{a_1, a_2, \ldots, a_m\}$$

which are the largest and smallest elements in these finite sets, respectively.

LECTURE 3: THE COMPLETENESS AXIOM

These are notes based upon the book "Real Analysis: A First Course", by Russel A. Gordon.

3.1. **Bounded and Unbounded Sets.** There are many ways to formulate what is known as "The Completeness Axiom". This axiom is what allows us to guarantee the existence of real roots. Also, it allows us to have convergent sequences converge to actual numbers. We will talk about this in the next section. The formulation for the axiom in your text is via sets with upper and lower bounds. Here's the definition:

(1) A nonempty set $S \subset \mathbb{R}$ is bounded from above if there is a number $M \in \mathbb{R}$ such that for all $x \in S$, $x \leq M$. The number M is called an upper bound.
(2) A nonempty set $S \subset \mathbb{R}$ is bounded from below if there is a number $m \in \mathbb{R}$ such that for all $x \in S$, $m \leq x$. The number m is called a lower bound.
(3) A nonempty set $S \subset \mathbb{R}$ is bounded if it is bounded from below and bounded from above. If so, there is a number $M \in \mathbb{R}$ such that for all $x \in S$, $|x| \leq M$. The number M is called a bound.
(4) A nonempty set $S \subset \mathbb{R}$ is unbounded if for all M, there is an $x \in S$ such that $M < |x|$.

Note that the definition for unbounded is the negation of the definition for bounded.

Examples: The interval $[0, 1]$ is bounded below and above. The set $[0, \infty)$ is bounded below but not above. The set $\mathbb{R}$ is unbounded.

The following definition is the most important one of this section:

(1) Suppose that S is bounded above. Then the **supremum** of S is the smallest upper bound of S(if it exists). This means that we call β the supremum of S if it is an upper bound of S AND every number less than β is not an upper bound. In this case, we write $\beta = \sup(S)$
(2) Suppose that S is bounded below. Then the **infimum** of S is the largest lower bound of S(if it exists). This means that we call α the infimum of S if it is a lower bound of S AND every number greater than α is not a lower bound. In this case, we write $\alpha = \inf(S)$.

3.1.1. *Sets of reals bounded above/below have sups/infs.* Based on our intuition of the real numbers, we certainly would believe that the set $(0, 1)$ has a supremum and an infimum. The infimum is 0 and the supremum is 1. What about the following sets? Are they bounded? Intuitively, what are the sups and infs, if they exist?

- $\{1/n : n \in \mathbb{N}\}$
- $\{x \in \mathbb{R} : x^2 \leq 4\}$
- $(0, \infty)$
- $[0, \infty)$

Important Note: A set which has a supremum or infimum need not contain that supremum or infimum.

3.1.2. *Subsets of the rationals bounded above/below need not have rational sups/infs.* In the text, the following set is discussed.

$$\{x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2\}$$

This set has a rational upper bound. For example, $3/2$ is a rational upper bound. If $x > 3/2$, then $x > 9/4 > 2$. It should be intuitively clear that the only number that it makes sense to say is $\sup(S)$ is $\sqrt{2}$. However, this number is not rational! For HW, please read the precise proof of this.

3.2. **Completeness Axiom.** The Completeness Axiom is the property of the reals that we need, along with the fact that $\mathbb{R}$ is an ordered field possessing a metric, to obtain the theorems of calculus. The completeness axiom is a consequence of the way in which the real numbers are constructed (by Cauchy sequences or Dedekind cuts). We take it as an AXIOM so that we can avoid performing the lengthy construction. In any event, it is an AXIOM in the sense that it is INDEPENDENT of the other stated properties of the real numbers. You cannot prove completeness simply from the fact that it is an ordered field possessing a metric.

The Completeness Axiom Every nonempty set of real numbers that is bounded above has a supremum.

Theorem 1 (Archimedean Property of the Reals)**.** *For all positive real numbers a and b, there exists a natural number n such that $na > b$.*

Proof. If $a > b$, let $n = 1$. Suppose $0 < a \leq b$. The proof is by contradiction. Suppose $na \leq b$ for all positive integers n. Let $S = \{na : n \text{ is a positive integer}\}$. Note that S is bounded above by b. By the Completeness Axiom, S has a supremum. Let $\beta = \sup(S)$.

Now, β is the *smallest* upper bound, Since $\beta - a < \beta$, we have that $\beta - a$ is not an upper bound of S. Then there is some positive integer k such that $\beta - a < ka$. This means that $\beta < (k+1)a$, which is a contradiction. □

Corollary 2. *The following are equivalent (TFAE)*

1. *The Archimedean Property of the Reals.*
2. *The set of positive integers is not bounded from above.*
3. *For each real number x, there is an integer n such that $n \leq x < n + 1$.*
4. *For each positive real number x, there exists a positive integer such that $1/n \leq x$.*

Proof. Once again following the text, we will show that (2) $\implies$ (3) and (4) $\implies$ (1). The other implications will be assigned as HW problems.

(2) $\implies$ (3): Let x be a real number. The result is clear if $-1 \leq x \leq 1$. Suppose first that $x > 1$. The proof of this also requires that we use a special property of the natural numbers called the Well-Ordering Principle. This says that any nonempty subset of the natural numbers always has a least element. Let S be the set of positive integers which are greater than or equal to x. The set has at least one element because of (2). Thus, S has a least element n. Then one of $n - 1 < x < n$ and $n \leq x < n + 1$ must be true. Either way,

the result is proved.

If $x < -1$, look at $-x > 1$. Then we have an m such that $m - 1 \leq -x < m$ or equivalently, $-m < x \leq -m + 1$. Hence, one of $-m \leq x < -m + 1$ and $-m + 1 \leq x < -m + 2$ is true. This proves the theorem.

(4) $\implies$ (1): Let a and b be two positive real numbers. Consider the quotient a/b. This is a positive real number. By (4), we have that there is a positive integer such that $1/n < a/b$. Then $b < na$, as desired. □

What is wrong with the following argument for (2) $\implies$ (3)?

Proof by contradiction. Suppose that for every integer n, either $x < n$ or $x > n + 1$. Then either $\mathbb{Z}$ is bounded above by $x - 1$ or $-x$. Hence, there must be some integer n such that $n \leq x$ and $x \leq n + 1$

Theorem 3. *For any $x, y \in \mathbb{R}$, there is some rational and some irrational number between x and y.*

Proof. Suppose that $x < y$. Then $y - x > 0$. By the Archimedean Property of the reals, there is an n such that $1 < n(y - x) = ny - nx$. This means that there is an integer m between ny and nx i.e. $nx < m < ny$. The rational number m/n is the number for which we are looking.

Now we consider the irrational case. In this case look at $\sqrt{2}x < \sqrt{2}y$. Then $y\sqrt{2} - x\sqrt{2} > 0$. By the Archimedean property of the real numbers, there is an n such that $1 < n(y\sqrt{2} - x\sqrt{2})$. Then there is an integer m between $nx\sqrt{2}$ and $ny\sqrt{2}$. If it happens that $m = 0$, then there is an integer $k > 0$ such that $1/k < ny\sqrt{2}$. The number $1/(kn\sqrt{2})$ is irrational and satisfies the desired properties. □

3.3. **Some roots are real.** In the book, the author shows that 5 has a real cube root. The argument is interesting, but we will see a more general (and far easier!) method when we discuss the Intermediate Value Theorem. It is a consequence of the following theorem, which we will prove later.

Theorem 4. *Every polynomial of odd degree has at least one real root.*

However, the Intermediate Value Theorem is ultimately a consequence of the Completeness Axiom!

3.4. **All reals have decimal expansions.** We have already seen that all rational numbers have a finite or repeating decimal expansion. The fact that any real number has a decimal expansion is yet another consequence of the completeness axiom. Recall that a decimal expansion is of the following form:

$$e_m e_{m-1} \dots e_1 . d_1 d_2 d_3 \dots = \sum_{i=1}^{m} 10^i e_i + \sum_{j=1}^{\infty} \frac{d_j}{10^j}$$

where $0 \leq e_i \leq 9$ for $1 \leq i \leq m - 1$, $1 \leq e_m \leq 9$ and $0 \leq d_j \leq 9$ for all $1 \leq j < \infty$.

Theorem 5. *All real numbers have a decimal expansion.*

Proof. It is sufficient to show that if $0 \leq x \leq 1$, then x has a decimal expansion. Let d_1 be the largest integer, $0 \leq d_1 \leq 9$ such that:

$$\frac{d_1}{10} < x$$

Call the number on the left hand side of the inequality x_1. Let d_2 be that largest integer, $0 \leq d_2 \leq 9$ such that:

$$\frac{d_1}{10} + \frac{d_2}{100} < x$$

Call the number on the left hand side of the inequality x_2. Continue inductively. For d_{n-1}, define d_n to be the integer, $0 \leq d_n \leq 9$ such that:

$$\frac{d_1}{10} + \frac{d_2}{100} + \frac{d_{n-1}}{10^{n-1}} + \frac{d_n}{10^n} < x$$

Call the number on the left hand side of the inequality x_n. Let $S = \{x_n : 1 \leq n < \infty\}$. This set has x as an upper bound. By the Completeness Axiom, S has a supremum β. To prove the theorem, we need only show $x = \beta$. We already know that $\beta \leq x$. Suppose $\beta < x$.

The idea is to use this to contradict the maximality of the numbers x_n. There is a positive integer n such that $1/n < x - \beta$. Now, for all $m \geq 0$, $m < 10^m$. Thus, $1/10^n < x - \beta$. Consider the following:

$$x \leq \frac{d_1}{10} + \frac{d_2}{100} + \ldots + \frac{d_{n-1}}{10^{n-1}} + \frac{d_n}{10^n} + \frac{1}{10^n} < \beta + \frac{1}{10^n} < x$$

This is a contradiction. □

LECTURE 4: COUNTABLE AND UNCOUNTABLE SETS

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

0.1. **Warm-up.** (Hilbert's Hotel) You have a hotel with infinitely many rooms, all of which are full. A guest shows up looking for a room? How do you accommodate her? Assume the rooms are numbered $1, 2, 3, \ldots$. Tell the person in room 1 to move to room 2, in room 2 to move to room 3, etc. The new guest occupies room 1. Everyone is accomodated!

0.2. **Review.** Let A, B be sets. Recall that a function $f : A \to B$ is said to be *one-to-one* if for all $x, y \in A$, $x \neq y \implies f(x) \neq f(y)$. To show that a function is one-to-one, you need to show this exact property. The function $f(x) = x^2$ is one-to-one when $x \geq 0$, but not one-to-one on all of $\mathbb{R}$.

A function $f : A \to B$ is said to be *onto* B if for all $y \in B$, there is an $x \in A$ such that $f(x) = y$. To show that a function is onto, you must always show this property exactly. The function $f(x) = x^2$ is onto the nonnegative reals, but it is not onto the set of all reals.

0.3. **Finite, Countable, Uncountable.** Suppose you have a bag of marbles and a bag of dice. Also suppose that you have forgotten how to count. Do you have more marbles or more dice? Here is a strategy for showing this: Line up all of the marbles in the sand in a line. Now, take out one dice at a time and lay it next to a marble. If you run out of dice before you run out of marbles, then you have more marbles than dice. If you run out of marbles but you still have dice in your bag, then you have more marbles than dice. If every marble has a dice next to it (onto) and every dice corresponds to only one marble (one-to-one!), then you have as many marbles as dice.

This argument also works when you have sets which are not finite. First some definitions, then some examples.

- A *one-to-one correspondence* between two sets A and B is a function $f : A \to B$ which is one-to-one and onto.
- A set S is said to be *finite* if it is empty or if the elements of S can be put into a one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$ for some n.
- A S is said to be *infinite* if it is not finite.
- A set S is said to be *countably infinite* if it can be put into one-to-one correspondence with the set of positive integers.
- A set S is said to be *countable* if it is finite or countably infinite.
- A set S is said to be *uncountable* if it is not countable.

0.4. **Examples.** The set S of positive even integers is countably infinite. Let $f : S \to \mathbb{N}$ be defined by $f(x) = x/2$. This map is one-to-one and onto.

The set of integers is countably infinite. $f : \mathbb{Z} \to \mathbb{N}$:

$$f(z) = \begin{cases} 2z + 1 & z \geq 0 \\ 2|z| & z < 0 \end{cases}$$

This is the same idea as the following hotel problem: All of the infinitely many rooms are full and infinitely many guests show up seeking refuge. How do you accommodate all of the guests? Tell guest in room i to go to room $2i$, and assign the new guests to rooms $1, 3, 5, 7, \ldots$! Of course, this is not fun for the guest in room 10^6, as they will have to walk past a million rooms before they get to their new room.

A countably infinite set is infinite. It is sufficient to show that $\mathbb{N}$ is not finite. We know that the set of prime numbers is not finite (can you give a simple proof?). If $\mathbb{N}$ were finite, then the set of primes would be finite, which is a contradiction.

0.5. **Famous Theorems.**

Lemma 1. *If A and B are countably infinite then $A \times B$ is countably infinite.*

Sketch. Let p, q be distinct primes. We have $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Define $F : A \times B \to \mathbb{N}$ by:

$$F(a, b) = p^{f(a)} q^{g(b)}$$

This function is one-to-one by the Fundamental Theorem of Arithmetic. As this is an infinite subset of a countably infinite set, it is countably infinite (prove this as an exercise). □

Theorem 2. *$\mathbb{Q}$ is countably infinite.*

Sketch. If $\frac{a}{b} \in \mathbb{Q}$, we write it as an ordered pair (a, b). The set of such ordered pairs is countable (by the above lemma). Now, $\mathbb{Q}$ is just the set of reduced fractions. So this is an infinite subset of a countably infinite set. Therefore, it is countably infinite. □

Theorem 3. *$(0, 1)$ is uncountable. Hence, the real numbers are uncountable.*

Cantor Diagonalization Argument. Every real number has a decimal expansion. Suppose $(0, 1)$ is countable. We make a list $x_1, x_2, \ldots$.

$$x_i = .a_{i1}a_{i2}a_{i3}\ldots$$

$$\begin{aligned}
x_1 &= .a_{11}\ a_{12}\ a_{13}\ a_{14}\ldots \\
x_2 &= .a_{21}\ a_{22}\ a_{23}\ a_{24}\ldots \\
x_3 &= .a_{31}\ a_{32}\ a_{33}\ a_{34}\ldots \\
x_4 &= .a_{41}\ a_{42}\ a_{43}\ a_{44}\ldots \\
\vdots &= \vdots
\end{aligned}$$

We define $b = .b_1 b_2 \ldots$ by:

$$b_1 = \begin{cases} 1 & \text{if } a_{11} \neq 1 \\ 2 & \text{if } a_{11} = 1 \end{cases}, \quad b_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$$

We must argue that b is not on the list. Suppose that b is equal to x_i for some i. The only way that this can happen is if $b_i = x_{ij}$ for $1 \leq j \leq i - 1$. By construction, $b_i \neq x_{ii}$. Decimal expansions of a real number fail to be unique only when there are an infinite number of successive nines (or zeros) in them (prove this as an exercise). Therefore, we must have $x_{ii} = 1$ and $x_{ij} = 9$ for all $j > i$. But b_{i+1} is either one or two. This is a contradiction, so b is not on the list and the real numbers are uncountable. □

LECTURE 5: REAL VALUED FUNCTIONS

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

5.1. **Cantor Set.** The Cantor set and Cantor function are important examples in real analysis. They are only briefly covered in the text, but they really should not be skipped!

Here is how the Cantor set is defined. First, it is a subset of $C_0 = [0,1]$. The Cantor set is defined by iteratively removing intervals. Define C_1 to be the set obtained from C_0 by removing the interval $(1/3, 2/3)$. The set $C_1 = [0,1/3] \cup [2/3,1]$. Define C_2 to be the set which is obtained from C_1 by removing the middle third of each of these intervals. This leaves you with $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. Continue this process to define C_i for each i. The Cantor set is $C = \cap_{i=0}^{\infty} C_i$.

Let us characterize the points that remain. As we have seen, every real number has a decimal expansion. The same argument can be used to show that every real number has a ternary expansion. For every x, $0 \leq x \leq 1$, this means

$$x = \frac{x_1}{3} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \dots$$

where $0 \leq x_i \leq 2$ for all i. In the construction of C_1, we removed all the points which have a ternary expansion with $x_1 = 1$ (Note that $1/3$ has a ternary expansion containing no ones, since $.1 = .0222...$ in base 3). In the construction of C_n, we removed all the points that have $x_n = 1$. Thus, the Cantor set has no points which have a 1 in the decimal expansion. In fact, we have taken out nothing else either. So these points completely characterize the Cantor set: no 1's in the ternary expansion.

From here, you set up a bijection to the interval $[0,1]$. You send an element of the ternary expansion to the binary expansion of a number in $[0,1]$:

$$\frac{x_1}{3} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \dots \to \frac{x_1/2}{2} + \frac{x_2/2}{2^2} + \frac{x_3/2}{2^3} + \dots$$

This means that the Cantor set is uncountable.

Now, how much of the set $[0,1]$ did we remove? First we removed $1/3$. Then we removed $2/9$. Then we removed $4/27$. The total "length" of points removed is:

$$\begin{aligned} \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots &= \frac{1}{3}\left(1 + \frac{2}{3} + \frac{4}{9} + \dots\right) \\ &= \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k \\ &= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} \\ &= 1 \end{aligned}$$

Thus, the Cantor set is an uncountable set which has no "length". As a point of contrast, the uncountable set of irrational number can be shown to have "length" 1.

5.2. **Cantor Ternary Function.** The Cantor ternary function is an example of a continuous real valued function on $[0,1]$ that is increasing but is constant except on a set of "length" (or *measure*) zero. Moreover, the derivative of the Cantor ternary function is zero *almost everywhere* but is not constant.

The Cantor ternary function is defined as follows. If x is in the the Cantor set, write out it's ternary expansion and define $c(x)$ to be:

$$c\left(\frac{x_1}{3}+\frac{x_2}{3^2}+\frac{x_3}{3^3}+\ldots\right)=\frac{x_1/2}{2}+\frac{x_2/2}{2^2}+\frac{x_3/2}{2^3}+\ldots$$

If x is not in the Cantor set, let N be the smallest natural number such that x_i from the ternary expansion is 1. Define:

$$c\left(x\right)=\frac{1}{2^N}+\sum_{i=1}^{N-1}\frac{x_i/2}{2^i}$$

In class we looked at a picture of the Cantor ternary function from the internet.

LECTURE 6: SEQUENCES AND CONVERGENCE

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

6.1. **Sequences and Convergence.** I asked you to read the definition of sequence and the basic properties of sequences for HW. These should have been review. The concepts are analogous to those that we have already seen. We start right away with the definition of convergence.

Definition: Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers. We write $\lim_{n\to\infty} x_n = L$ if for all $\varepsilon > 0$, there is an $N > 0$ such that if $n \geq N$, $|x_n - L| < \varepsilon$. If a sequence has a limit, we say that it converges.

A major focus of this course is in learning how to write proofs involving estimations. The definition of convergence for sequences implies that you will be writing estimation proofs when you prove that a sequence converges.

Example: Prove that:

$$\lim_{n\to\infty} \frac{3n+1}{2n+6} = \frac{3}{2}$$

First, always write: Let $\varepsilon > 0$. The trick is to start with what you want to know. We want to find an N such that if $n \geq N$, then:

$$\left|\frac{3n+1}{2n+6} - \frac{3}{2}\right| < \varepsilon.$$

$$\begin{aligned}\left|\frac{3n+1}{2n+6} - \frac{3}{2}\right| &= \left|\frac{2(3n+1) - 3(2n+6)}{2(2n+6)}\right| \\ &= \left|\frac{16}{2(2n+6)}\right| \\ &= \left|\frac{8}{(2n+6)}\right|.\end{aligned}$$

Now, $n < 2n + 6$ for all $n > 0$. This gives:

$$\left|\frac{3n+1}{2n+6} - \frac{3}{2}\right| = \left|\frac{8}{(2n+6)}\right| < \frac{8}{n}.$$

If we choose $N = 8/\varepsilon$, we have the desired inequality. Now we are ready to write out the proof.

Proof. Let $\epsilon > 0$ be given. Let $N = 8/\varepsilon$. Then if $n \geq N$, we have $\frac{8}{n} \leq \varepsilon$.

$$\left|\frac{3n+1}{2n+6} - \frac{3}{2}\right| = \left|\frac{8}{(2n+6)}\right| < \frac{8}{n} \leq \varepsilon$$

This proves the statement, by the definition of convergence. □

Example: Prove that:

$$\lim_{n\to\infty} \frac{1-n}{7n+1} = -\frac{1}{7}$$

Let $\epsilon > 0$. Start with the end.

$$\begin{aligned}\left|\frac{1-n}{7n+1}+\frac{1}{7}\right| &= \left|\frac{7-7n+7n+1}{7(7n+1)}\right| \\ &= \left|\frac{8}{49n+7}\right|\end{aligned}$$

Note that for $n > 0$, $n < 49n + 7$. Then we have:

$$\left|\frac{1-n}{7n+1}+\frac{1}{7}\right| < \frac{8}{n}$$

We can choose $N = 8/\varepsilon$ so that we get our desired result. Now we write our proof.

Proof. Let $\varepsilon > 0$. Choose $N = 8/\varepsilon$. Then if $n \geq N$, we have $\varepsilon \geq 8/n$. This gives:

$$\left|\frac{1-n}{7n+1}+\frac{1}{7}\right| < \frac{8}{n} \leq \varepsilon$$

Hence, we have the desired result. □

6.2. **Properties of convergent sequences.**

Theorem 1. *If a sequence converges, then its limit is unique. In other words, if $x_n \to L$ and $x_n \to M$, then $L = M$.*

Proof. We show that $|L - M| < \varepsilon$ for all $\varepsilon > 0$. Since $x_n \to L$ and $x_n \to M$, there is an $N_L > 0$ such that for all $n \geq N_L$, $|x_n - L| < \frac{\varepsilon}{2}$ and an $N_M > 0$ such that for all $n \geq N_M$, $|x_n - M| < \frac{\varepsilon}{2}$. Let $n \geq \max\{N_M, N_L\}$. Then:

$$|L - M| = |L - x_n + x_n + M| \leq |L - x_n| + |M - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Theorem 2. *A convergent sequence is bounded.*

Proof. Suppose $x_n \to L$. There exists an N such that if $n \geq N$, then $|x_n - L| < 1$. In particular, we have that $L - 1 < x_n < L + 1$. Then for $n \geq N$, x_n is bounded below by $L - 1$ and above by $L + 1$. The terms $x_0, x_1, \ldots, x_N$ are bounded as they are a finite set. A lower bound for the sequence is $\min\{x_1, \ldots, x_N, L - 1\}$. an upper bound is $\max\{x_1, \ldots, x_N, L + 1\}$. □

6.3. **How to show that a sequence does not converge.** To show that a sequence does not converge, we use the negation of the definition of convergence.

Negation of the Convergence Definition: A sequence $\{x_n\}$ does not converge to L if there is an $\varepsilon > 0$ such that for all $N > 0$, there is an $n \geq N$ such that $|x_n - L| \geq \varepsilon$.

Example: Prove that the sequence $\{(-1)^n\}_{n=0}^{\infty}$ does not converge to any real number.

It is easy to see how this works once you've looked at $L = 1$. Let $\varepsilon = 1$. Let $N > 0$ be given. Then $|x_n - 1| = 2 > 1$ for infinitely many values of n. The general argument follows similarly.

Example: Prove that the sequence $\{2^n\}$ does not converge. This one is easy. The sequence is not bounded, so it doesn't converge.

Definition A sequence $\{x_n\}$ converges to ∞ if for all $M > 0$, there is an $N > 0$ such that if $n > N$, then $x_n > M$. The sequence converges to $-\infty$ if $\{-x_n\}$ converges to ∞.

Example: $2^n \to \infty$. $-2^n \to -\infty$, $(-2)^n$ does not converge.

6.4. **Using known results about convergence to prove convergence.**

Theorem 3. *Suppose that $a_n \to a$ and $b_n \to b$. Then we have:*

(1) *$ca_n \to ca$, where c is any constant.*
(2) *$a_n \pm b_n \to a \pm b$.*
(3) *$a_n b_n \to ab$.*
(4) *$a_n/b_n \to a/b$, as long as $b_n \neq 0$ for all n and $b \neq 0$.*

Proof. We prove (2), (3), (4). For (2), just consider the + case. Let $\varepsilon > 0$ be given. There is an $N_1 > 0$ such that if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2}$. There is an $N_2 > 0$ such that if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then:

$$\begin{aligned} |a_n + b_n - (a+b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

For (3), we do the add-and-subtract trick.

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n||b_n - b| + |b||a_n - a|. \end{aligned}$$

We know that $|a_n|$ is bounded so, $|a_n| \leq A$ for some $A > 0$. Similarly, $|b| \leq B$ for some $B > 0$. Choose N_1 so large that $|a_n - a| < \frac{\varepsilon}{2B}$ when $n \geq N_1$. Choose N_2 so large that $|b_n - b| < \frac{\varepsilon}{2A}$ when $n \geq N_2$. Then we have:

$$|a_n b_n - ab| < \varepsilon.$$

For (4), we want to analyze the sequence $|a_n/b_n - a/b|$:

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - b_n a}{b_n b}\right| = \left|\frac{a_n b - a_n b_n + a_n b_n - b_n a}{b_n b}\right|.$$

Note that the sequence a_n is bounded above by a positive number, $|a_n| \leq A$ and the sequence $b_n b$ is bounded below by a positive number, $B \leq |b_n b|$, for all n. This gives:

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{A}{B}|b - b_n| + \frac{1}{|b|}|a_n - a|.$$

We have complete control over the expression on the right, so we can show that the expression on the left is $< \varepsilon$. □

The following result is one of the most useful for showing that a limit converges or finding the value of that limit.

Theorem 4 (The Squeeze Theorem)**.** *Suppose that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences of real numbers. If $a_n \leq b_n \leq c_n$ for every n and $a_n \to L$ and $c_n \to L$, then $b_n \to L$.*

Proof. Consider the following estimation:

$$|b_n - L| = |b_n - a_n + a_n - L| \leq |b_n - a_n| + |a_n - L| \leq |c_n - a_n| + |a_n - L|.$$

We know that $\lim_{n\to\infty}(c_n - a_n) = 0$. Then there is an N_1 such that if $n > N_1$ such that if $n \geq N_1$, $|c_n - a_n| < \varepsilon/2$. There is an N_2 such that if $n \geq N_2$, then $|a_n - L| < \varepsilon/2$. Then:

$$|b_n - L| < \varepsilon.$$

as long as $n > \max\{N_1, N_2\}$. □

Example: The canonical example is $\{\sin(n)/n\}$. Use the Squeeze Theorem to show that it converges to 0.

Theorem 5. *Let $\{x_n\}$ be a sequence of points in $[a, b]$. If $x_n \to x$, then $x \in [a, b]$.*

Proof. To prove this it is sufficient to show that if c_n is a sequence with $c_n \geq 0$ for all n and $c_n \to c$, then $c \geq 0$. If $c < 0$, we have $|c_n - c| = |c_n| + |c| \geq |c|$ for all n. Since $c_n \to c$, there is an $N > 0$ such that if $n \geq N$, then $|c_n - c| < |c|$. This is a contradiction.

Now, we have $b - x_n \geq 0$ and $x_n - a \geq 0$ for all n. The above observation tells us that $x - a \geq 0$ and $b - x \geq 0$. This completes the proof. □

LECTURE 7: CAUCHY SEQUENCES AND MONOTONE SEQUENCES

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

7.1. **Bounded Monotone Sequences.** This section presents an important technique for determining when a sequence converges in the event that it is difficult or impossible to find the limiting value.

Theorem 1. *A monotone sequence converges if and only if it is bounded.*

Proof. Any convergent sequence is bounded. Suppose that we have a bounded increasing sequence $\{a_n\}$. Let A be the set of points of the sequence. This is a bounded set. By the Completeness Axiom, A has a supremum a.

We claim that the sequence converges to a. Let $\varepsilon > 0$ be given. By definition of the supremum, there is an N such that $a - \varepsilon < a_N$. Since the sequence is increasing, $a - \varepsilon < a_n$ for all $n \geq N$. Thus, $|a_n - a| < \varepsilon$ for all $n \geq N$. This proves the theorem. □

Example:(from text) Show that the sequence below converges:

$$\left\{\sum_{k=1}^{n} \frac{1}{k2^k}\right\}_{n=1}^{\infty}$$

It is pretty clear that the sequence is increasing. It is bounded by 1 since:

$$\sum_{k=1}^{n} \frac{1}{k2^k} < \sum_{k=1}^{n} \frac{1}{2^k} = -1 + \sum_{k=0}^{n} \frac{1}{2^k} = -1 + \frac{1 - \left(\frac{1}{2^{n+1}}\right)}{1/2} = 1 - \frac{1}{2^n} < 1$$

It is hard to find out to what this sequence converges. In fact, if we remember everything that we know about power series, we can show that it is equal to $\ln(2)$. Recall that if $|x| < 1$,

$$\frac{d}{dx} \sum_{n=1}^{\infty} \frac{1}{k} x^k = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

$$\int \frac{1}{1-x} dx = -\ln|1-x| + C = \sum_{n=1}^{\infty} \frac{1}{k} x^k$$

This means that $C = 0$. Plugging in $x = 1/2$ gives the desired result.

Example: Define a sequence recursively by $a_0 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$. Show that this sequence converges and find its limit. Note that the limit of the sequence may be written like this:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$$

To show that it is convergent, we show that it is increasing and that it is bounded. We know that the square root function is increasing. If $x < y$, then $\sqrt{x} < \sqrt{y}$.

First we show that the sequence is bounded above by 2. It is certainly true that $a_0 < 2$. We proceed by induction. Suppose that $a_i < 2$ for $1 \leq i \leq n$. We will show that $a_{n+1} < 2$. Since $a_n < 2$, we have $2 + a_n < 4$ and $a_{n+1} = \sqrt{2 + a_n} < 2$. Thus, by the principle of mathematical induction, the sequence is bounded above by 2. It is clear that the sequence is bounded below by 0.

Now we show that the sequence is increasing. We need $a_n < a_{n+1}$. To do this, we must show that $a_n < \sqrt{2 + a_n}$. This is equivalent to:

$$a_n^2 - a_n - 2 < 0$$

This can only happen in $a_n \in (-1, 2)$ for all n. We know that this is true already, since we have established that the sequence is bounded below by 0 and above by 2.

The above theorem shows that the sequence converges. To find the limit L, we write:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}} = L$$

From this we get the equation $L^2 - 2 = L$. The only roots are $2, -1$. The only possibility is $L = 2$. This gives the satisfying equation:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}} = 2$$

7.2. **Cauchy Sequences.** Another important technique for showing that sequences converge is to use Cauchy sequences.

Definition A sequence $\{a_n\}$ is said to be a Cauchy sequence if for all $\varepsilon > 0$, there is an N such that if $n, m \geq N$, then:

$$|a_n - a_m| < \varepsilon$$

Example: Show that the following sequence is a Cauchy sequence:

$$a_n = \frac{1}{n}$$

Let $\varepsilon > 0$ be given. Choose N so that $1/N < \varepsilon$. Let $n, m \geq N$, $m > n$:

$$\left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{n} < \frac{1}{N} < \varepsilon$$

Thus, this sequence is a Cauchy sequence.

Example:(from text) Show that the following sequence is a Cauchy sequence:

$$\left\{ \sum_{k=1}^{n} \frac{1}{k^2} \right\}_{n=1}^{\infty}$$

Suppose that $m < n$. Then we have:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{m} \frac{1}{k^2} &= \sum_{k=m+1}^{n} \frac{1}{k^2} \\
&< \sum_{k=m+1}^{n} \frac{1}{k(k-1)} \\
&= \sum_{k=m+1}^{n} \frac{1}{k-1} - \frac{1}{k} \qquad \text{(partial fractions)} \\
&= \frac{1}{m} - \frac{1}{n} \\
&< \frac{1}{m} \\
&< \frac{1}{N} \\
&< \varepsilon
\end{aligned}$$

So choosing $N > 1/\varepsilon$ shows that the sequence is Cauchy.

Theorem 2. *If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is bounded.*

Proof. There is an N such that if $n, m \geq N$, then $|a_n - a_m| < 1$. Then for all $n \geq N$, we have $n \geq N$, $a_n \in (a_N - 1, a_N + 1)$. A lower bound for the sequence is $\min\{a_1, \ldots, a_{N-1}, a_N - 1\}$. An upper bound is $\max\{a_1, \ldots, a_{N-1}, a_N + 1\}$. □

Theorem 3. *A sequence $\{x_n\}$ of real numbers converges if and only if it is Cauchy.*

Proof. It is easy to show that a convergent sequence is Cauchy. To prove the converse, we must show that if a sequence is Cauchy, it converges. To do this, we need to specify the limit. We know that we can find limits of bounded monotone sequences. We construct a bounded monotone sequence of our Cauchy sequence. Define:

$$a_n = \inf\{x_{n+1}, x_{n+2}, \ldots\}$$

We have $a_n \leq a_{n+1}$ since the infimum is taken over a smaller set. This sequence is bounded because our original Cauchy sequence is bounded. Thus we know that this sequence has a limit $a_n \to a$.

The goal is to show that a is also the limit of x_n. To do this, we must compare how close the sequence $\{a_n\}$ is to the sequence $\{x_n\}$ for large N. Now, since $\{x_n\}$ is Cauchy, there is an $N_1 > 0$ such that if $n, m \geq N_1$, then $|x_n - x_m| < \varepsilon/3$. Therefore, if $n, m \geq N_1$, we must have that $|a_n - x_m| \leq 2\varepsilon/3$ (think of the picture!). Now choose N_2 so large that if $n \geq N_2$, then $|a_n - a| < \varepsilon/3$. Choose an $n \geq \max\{N_1, N_2\}$. If $m \geq N_1$, we have:

$$|x_m - a| \leq |x_m - a_n| + |a_n - a| < \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon < \varepsilon$$

□

7.3. **Some Common Sequences.** We follow the text here and show that some common sequences converge:

Theorem 4. *The following sequences converge.*

1. *If $r > 1$, then the sequence r^n converges to ∞.*
2. *If $|r| < 1$, the sequence $\{r^n\}$ converges to 0.*
3. *If $|r| < 1$, the sequence $\{\sum_{k=0}^{n} r^k\}$ converges to $1/(1-r)$*
4. *The sequence $\sqrt[n]{n} \to 1$*
5. *The sequence $(1 + 1/n)^n$ converges to a number e.*
6. *The sequence $\sum_{k=0}^{n} 1/k!$ converges to e.*

Proof. For the first, we use Bernoulli's inequality:

$$r^n = (1 + (r-1))^n \geq 1 + n(r-1)$$

The sequence on the right clearly converges to ∞, so the sequence on the left is forced to as well.

For $|r| < 1$, we look at $1/r > 1$ when $r > 0$. Let $\varepsilon > 0$ be given. Since $(1/r)^n \to \infty$, we have that there is an N such that if $n \geq N$, then $1/r^n > 1/\varepsilon$. Then $r^n < \varepsilon$. Thus, $r^n \to 0$. Suppose $-1 < r < 0$. Consider the sequences $-|r|^n \leq r^n \leq |r|^n$. The result follows by the squeeze theorem.

For (3), we have the geometric sequence:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^n}{1 - r}$$

The result follows from the previous item.

For the moment, suppose that we know that all n-th roots of positive real numbers exists. We will prove this in the next chapter using the Intermediate Value Theorem. Following the text, let $t_n = \sqrt[n]{n} - 1$. Then

$$n = (t_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} t_n^k > \frac{n(n-1)}{2} t_n^2$$

when $n \geq 2$. Then rearranging, we see that:

$$0 \leq t_n \leq \sqrt{\frac{2}{n-1}}$$

By the Squeeze Theorem, we see that $t_n \to 0$. This proves (4).

Lastly, we establish the existence of the Euler's constant e. We use the following identity throughout.

$$\binom{n}{i} \frac{1}{n^i} = \frac{1}{i!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{i-1}{n}\right)$$

From this we see that:

$$\binom{n}{i} \frac{1}{n^i} \leq \binom{n+1}{i} \frac{1}{(n+1)^i}, \qquad \binom{n}{i} \frac{1}{n^i} \leq \frac{1}{i!}$$

We need only show that the sequence in (5) is bounded and increasing.

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} < \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(n+1)^k} = \left(1 + \frac{1}{n+1}\right)^{n+1}$$

This shows that the sequence in (5) is increasing. Now we must show that it is bounded. Note that $2^{k-1} \leq k!$ for all $k \geq 1$ (induction proof). Then:

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \leq \sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 1 + \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{2^k} = 3$$

The last sequence is covered as an exercise in the text. □

LECTURE 8: SUBSEQUENCES

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

8.1. **Subsequences.** Subsequences are very important in mathematics, specifically in the fields of topology and analysis.

Definition: Let $\{x_n\}$ be a sequence. A subsequence of $\{x_n\}_{n=0}^{\infty}$ is a sequence $\{x'_k\}_{k=0}^{\infty}$ where for each k, $x'_k = x_{n_k}$ for some n_k and the sequence of integers $\{n_k\}_{k=0}^{\infty}$ is strictly increasing.

Example: In general every subsequence has infinitely many subsequences. Identify 2 convergent subsequences of $\{(-1)^n\}$, i.e. two subsequences which are convergent sequences.

Example: Find two different increasing subsequence of the sequence $\{(-1)^n 2^{(-1)^n n}\}_{n=0}^{\infty}$. Find increasing and decreasing subsequences of the sequence $\{(-1)^n/n^2\}_{n=1}^{\infty}$.

Theorem 1. *Let $\{x_n\}$ be a sequence of real numbers.*

(1) *If $x_n \to L$ then every subsequence converges to L.*
(2) *If $\{x_n\}$ has two subsequences which converge to different values, then $\{x_n\}$ does not converge.*

Proof. For the first fact, let n_k be a strictly increasing sequence of positive integers. Let $\varepsilon > 0$. Since $x_n \to L$, there is an $N > 0$ Such that if $n \geq N$, then $|x_n - L| < \varepsilon$. Note that $n_k \geq k$ for all k. Thus, if $k \geq N$, we have $n_k \geq N$. Hence, $|x_{n_k} - L| < \varepsilon$.

Suppose that $x'_k \to L$, $x''_k \to M$, and $L > M$. Let $\varepsilon = (L - M)/4$. There exists an N_L such that if $k \geq N_L$, then $|x'_k - L| < \varepsilon$. There exists an N_M such that if $k \geq N_M$, then $|x''_k - M| < \varepsilon$. Then for $k \geq \max\{N_L, N_M\}$, we have that

$$|x'_k - x''_k| \geq \frac{L - M}{2}$$

Thus, the original sequence cannot be Cauchy, and we're done. □

Theorem 2. *Every sequence of real numbers has a monotone subsequence.*

Proof. The proof follows the one from the text (it is a well-known argument). Let $\{x_n\}$ be a sequence of real numbers. Let S be the set of positive integers such that x_n is a lower bound for the sequence $\{x_{n+1}, x_{n+2}, \ldots\}$. The set S is either infinite or finite. If it is infinite, we have $x_{n_k} \leq x_{n_l}$ whenever $n_k < n_l$ are elements of S. This is an increasing subsequence.

Otherwise, the set S is finite. Therefore, there is an integer n_1 which is larger than every element of S. If $k \geq n_1$, we have that k is not in S. This means that there is an integer $n_2 > n_1$ such that $x_{n_2} < x_{n_1}$. This process can be continued inductively to produce a sequence of strictly increasing positive integers such that $x_{n_{k+1}} \leq x_{n_k}$ for all k. In other words, we have produced a decreasing (hence monotone) sequence. □

Theorem 3 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. This is immediate from the previous theorem. A bounded sequence has a monotone subsequence which is bounded. A bounded monotone sequence converges. □

8.2. **Limit Inferior and Limit Superior.**

Definition: Let $\{x_n\}$ be a sequence of real numbers.

(1) If the sequence is bounded below, the limit inferior of $\{x_n\}$ is defined to be:

$$\liminf x_n = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, \ldots\}$$

If it is not bounded below, we define $\liminf x_n = -\infty$.

(2) If the sequence is bounded above, the limit superior of $\{x_n\}$ is defined to be:

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, \ldots\}$$

If it is not bounded above, we define $\limsup x_n = \infty$.

Note that lim inf and lim sup always exist for any sequence of real numbers. If a sequence is appropriately bounded, then lim inf and lim sup are limits of increasing and decreasing sequences, respectively.

Example: Compute lim inf and lim sup of the following sequences.

(1) $a_n = (-1)^n$.
(2) $b_n = (-1)^n 2^{(-1)^n n}$
(3) $c_n = n^2$
(4) $d_n = (-1)^n/n^2$

Theorem 4. *(from text) Let $\{x_n\}$ be a bounded sequence of real numbers.*

(1) *If $m \le x_n \le M$ for all n, then $m \le \liminf x_n \le \limsup x_n \le M$.*
(2) *If $\beta > \limsup x_n$, then there is an $N > 0$ such that $\beta > x_n$ for all $n > N$.*
(3) *If $\alpha < \liminf x_n$, then there is an $N > 0$ such that $\alpha < x_n$ for all $n > N$.*
(4) *The sequence converges if and only if $\liminf x_n = \limsup x_n$.*
(5) *The following equations hold for every real number c:*

$$\liminf(c + x_n) = c + \liminf x_n, \quad \limsup(c + x_n) = c + \limsup(x_n)$$

(6) *For $c > 0$, the following equations hold:*

$$\liminf c \cdot x_n = c \cdot \liminf x_n, \quad \limsup c \cdot x_n = c \cdot \limsup x_n$$

(7) *For $c < 0$ the following equations hold:*

$$\liminf c \cdot x_n = c \cdot \limsup x_n, \quad \liminf c \cdot x_n = c \cdot \liminf x_n$$

(8) *If $\{x_{p_n}\}$ is a subsequence, we have the inequalities:*

$$\liminf x_n \le \liminf x_{p_n} \le \limsup x_{p_n} \le \limsup x_n$$

(9) *For all $\varepsilon > 0$, the sets $\{n : -\varepsilon + \limsup x_n < x_n\}$ and $\{n : \varepsilon + \liminf x_n > x_n\}$ are infinite.*

Proof. Most of the properties were discussed briefly in class. We'll prove (2) and (4) only.

Let $L = \limsup x_n$ and $a_n = \sup\{x_{n+1}, \ldots\}$. We have $L < \beta$. Choose ε such that $L + \varepsilon < \beta$. There is an $N > 0$ such that if $n > N$, then $a_n \in (L - \varepsilon, L + \varepsilon)$ for all $n \ge N$. Since $a_n \ge x_k$ for all $k > n$, we must have $x_k \le a_n < \beta$ for all $k > N$.

If $x_n \to L$, let $\varepsilon > 0$ be given. Then there is an $N > 0$ such that if $n \ge N$, then $x_n \in (L - \varepsilon, L + \varepsilon)$. Therefore, $\inf\{a_{n+1}, a_{n+2}, \ldots\} \ge L - \varepsilon$ for all $n \ge N$. Then $\liminf x_n \ge L - \varepsilon$. Similarly, we can show that $\limsup x_n \le L + \varepsilon$. This gives the inequality:

$$L - \varepsilon \le \liminf x_n \le \limsup x_n \le L + \varepsilon$$

Since this is true for every $\varepsilon > 0$, we must have that the two are equal to one another.

Conversely, suppose that $L = \limsup x_n = \liminf x_n < \infty$. Let $\varepsilon > 0$ be given. Since $L - \varepsilon <$

$\liminf x_n$, there is an N_1 such that if $n > N_1$, then $L - \varepsilon < x_n$. Similarly, there is an N_2 such that if $n \geq N_2$, then $x_n < L+\varepsilon$. If we choose $N = \max\{N_1, N_2\}$, we have that $n \geq N$ implies $|x_n - L| < \varepsilon$.

For the last part, Let S be the first set and $L = \limsup x_n$. Suppose that it is finite. Let $N = \max(S)$. Then $\sup\{x_n, \ldots\} \leq L - \varepsilon$ for all $n > N$. But this is implies that $L \leq L - \epsilon$, which is a contradiction. □

Definition: If $\{x_n\}$ is a bounded sequence containing a subsequence which converges to a real number L, then L is said to be a subsequential limit of $\{x_n\}$.

Theorem 5. *(from text) Let $\{x_n\}$ be a bounded sequence and let S be the set of all subsequential limits of $\{x_n\}$. Then S contains its infimum and supremum and $\liminf x_n = \inf S$ and $\limsup x_n = \sup S$. In particular, there is a subsequence of $\{x_n\}$ which converges to the $\liminf$ and and a subsequence which converges to $\limsup$.*

Proof. It is clear that S must be bounded. This means that S has a supremum and an infimum. First we show that $\limsup x_n \geq \sup S$. Let $s \in S$. There is a subsequence $x_{n_k} \to s$. Then:

$$s = \lim_{n\to\infty} x_{n_k} = \limsup x_{n_k} \leq \limsup x_n$$

Therefore, $\sup(S) \leq \limsup x_n$.

Now we show that there is a subsequence of $\{x_n\}$ which converges to $\limsup x_n$. Let $N > 0$ be given. The set below is infinite:

$$A_N = \{n : -1/N + \limsup x_k < x_n < 1/N + \limsup x_k\}$$

It is infinite because Theorem 4 guarantees the following:

(1) that for all but finitely many n, $x_n < 1/N + \limsup x_k$ and,
(2) there are infinitely many n such that $-1/N + \limsup x_k < x_n$.

Choose $n_1 \in A_1$. Choose $n_2 \in A_2$, $n_2 > n_1$. Repeating this process gives a subseqence $x_{n_k} \to \limsup x_{n_k}$ □

Example: There is a sequence of real numbers in $[0, 1]$ for which every real number in $[0, 1]$ is a subsequential limit.

To see this, first let us consider the set A of rational numbers in $[0, 1]$. If $a \in A$, the sequence $x_n = a - \frac{1}{n}$ is a sequence of rational numbers which converges to a. Recall that the set of rational numbers is countable. We label each a in A by an index i corresponding to this enumeration and the corresponding sequence x_{in}. Consider the function $f(2^i 3^n) = x_{in}$. Since the set of pairs (i, n) is countable, we get a well defined function from the natural numbers to the set of x_{in}. This is a sequence.

For fixed i, we have a subsequence $f(2^i 3^n) \to a_i$. Thus, every rational number is a limit point of this sequence. This is an example of a sequence that has at least countably many limit points. We now show that it has uncountably many limit points.

Let $r \in (0, 1)$ be irrational. We know that there is a sequence q_k of rational numbers, $q_k \to r$. For each q_k, there is an i_k such that $x_{i_k n} \to q_k$ as $n \to \infty$. Choose an n_k such that $|x_{i_k n_k} - q_k| < 1/k$. There is a subsequence corresponding to the rational numbers $x_{i_k n_k}$. It is easy to see that $a_{i_k n_k} \to r$ as $k \to \infty$. Thus, r is a subsequential limit of our sequence.

8.3. **Some problems and their solutions.** Students should do 3, 9, 13,16,19,32. Challenge: (2.4 9,12)

(#8 from text 2.3) Let $x_1 = 1$ and let $x_n = (1 + x_n)^{-1}$ for all $n \geq 2$. Find a recursive definition for $a_n = x_{2n}$. Show that $a_n^2 + a_n < 1$ for all n and that $\{a_n\}$ is increasing. For each n, let $b_n = x_{2n-1}$. Prove that b_n is decreasing. Prove that the sequence $\{x_n\}$ converges and find the limit.

Proof. This is a very instructive problem. It is easy to see that:

$$a_n = \frac{1 + a_{n-1}}{2 + a_{n-1}}, a_0 = 0$$

is the desired recursive relation. We show by induction that the $a_n^2 + a_n < 1$. It is true for $n = 0$. Now, suppose it is true up to $n - 1$. Then we have:

$$\begin{aligned} a_{n-1}^2 + a_{n-1} &< 1 \\ 3 + 2a_{n-1}^2 + 5a_{n-1} &< 4 + a_{n-1}^2 + 4a_{n-1} \\ (1 + a_{n-1})(2a_{n-1} + 3) &< (2 + a_{n-1})^2 \\ \frac{(1 + a_{n-1})(a_{n-1} + 1 + 2 + a_{n-1})}{(2 + a_{n-1})^2} &< 1 \\ a_n^2 + a_n &< 1 \end{aligned}$$

So the inequality holds for all n. The sequence a_n is increasing because:

$$\begin{aligned} a_n^2 + a_n &< 1 \\ 2a_n + a_n^2 &< 1 + a_n \\ a_n &< \frac{1 + a_n}{2 + a_n} = a_{n+1} \end{aligned}$$

Since $a_n \leq a_n^2 + a_n < 1$, this is a bounded increasing sequence and hence has a limit. Similarly, we can show that $b_n = (1 + b_{n-1})/(2 + b_{n-1})$, $b_1 = 1$, $b_n^2 + b_n > 1$, and the sequence $\{b_n\}$ is increasing. Both sequences have the same limit: $L = (-1 + \sqrt{5})/2$. From this, one can conclude that $x_n \to L$. □

(#13 from text 2.3) Let $\{x_n\}$ be a bounded sequence and suppose that every convergent subsequence converges to L. Prove that $x_n \to L$.

Proof. We know that the set S of subsequential limits of L is nonempty. We also have that there is a subsequence that converges to the $\liminf$ and a subsequence which converges to the $\limsup$. Since both of these converge to L, and hence $\limsup x_n = \liminf x_n$, we have by a previous proposition that $x_n \to L$. □

(#16 from text 2.3) Prove the following statement about Cauchy sequences without using the fact that a Cauchy sequence of real numbers converges: If a subsequence of a Cauchy sequence converges, then the Cauchy sequence converges.

Proof. Let L be the limit of the subsequence $\{x_{n_k}\}$ and let $\varepsilon > 0$ be given. There is an $K > 0$ such that if $k \geq K$, then $|x_{n_k} - L| < \varepsilon/2$. There is an $N > 0$ such that if $n, m \geq N$, then $|x_n - x_m| < \varepsilon/2$. Recall that $n_k \geq k$ for all $k \geq 1$. If $j \geq \max\{N, K\}$, we have:

$$|x_j - L| = |x_j - x_{n_j} + x_{n_j} - L| \leq |x_n - x_{n_j}| + |x_{n_j} - L| < \varepsilon$$

This proves the statement. □

(#12 from text 2.4) let $\{x_n\}$ be a sequence of positive numbers and suppose that the sequence $\{x_{n+1}/x_n\}$ converges to L. If $L < 1$, show that $x_n \to 0$.

Proof. Choose $\nu > 0$ so that $L + \nu < 1$. There is an $N > 0$ such that if $n \geq N$,

$$L - \nu < \frac{x_{n+1}}{x_n} < L + \nu$$

Note that we may write x_n as a product of these ratios:

$$x_n = x_N \cdot \prod_{k=N+1}^{n} \frac{x_k}{x_{k-1}}$$

Therefore, if $n \geq N$, we have:

$$x_n \leq (L+\nu)^{n-N} x_N = (L+\nu)^n \frac{x_N}{(L+\nu)^N}$$

Now, let $\varepsilon > 0$ be given. Choose M_N such that if $m \geq M_N$, then:

$$(L+\nu)^m < \frac{\varepsilon}{x_N/(L+\nu)^N}$$

If $n \geq \max\{N, M_N\}$, then $x_n < \varepsilon$ and we conclude that $x_n \to 0$. □

LECTURE 9: THE LIMIT OF A FUNCTION

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

9.1. **Definition of the Limit, Copious Examples.** In this section, we define the limit of a function and go through numerous examples of the $\epsilon - \delta$ type proof.

Definition Let $f(x)$ be a function defined on every point in an open interval I except possibly at a point $a \in I$. We say that $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Example: As a start, let us look at $f(x) = x^2$, $a = 2$. The hypotheses are met. Our experience tells us that $L = 4$. We want to prove this from the definition of the limit. The game is this: A ε is given, you find a δ satisfying the hypotheses. Suppose $\varepsilon = 1/4$. It is your job to find a δ so that if $0 < |x - 2| < \delta$, then $|x^2 - 4| < \frac{1}{4}$.

Draw a picture. How close to 2 must x be so that $f(x)$ is within .25 of 4? For $f(x)$ to be within $1/4$ of 4, we need $3.75 < f(x) < 4.25$. Looking at a graph, we see that x must be between $\sqrt{3.75} \approx 1.936 < 1.94$ and $\sqrt{4.25} \approx 2.061 > 2.06$. Therefore, as long as x is within .06 of 2, all of the $f(x)$ values are within .25 of 4. Another way to say this is that as long as:

$$0 < |x - 2| < .06$$

we will always have:

$$|f(x) - 4| < .25$$

Of course, we want to know this for ANY $\varepsilon > 0$. Before we do this, let's look at an easier problem.

Example: Using the definition of the limit, prove that:

$$\lim_{x \to 1} 2x + 1 = 3$$

Example: Using the definition of the limit, prove that:

$$\lim_{x \to 2} x^2 = 4$$

It is always a good idea to keep in mind what you can control and what you can't control. You can control the distance from x to 2. That is the only thing you can control. Start at the end:

$$|x^2 - 4| = |x - 2||x + 2|$$

We can control $|x - 2|$. Suppose that $|x - 2| < 1$. Then $1 < x < 3$. We can use this to control the size of $|x + 2|$. Clearly, $3 < x + 2 < 5$. Hence if $|x - 2| < 1$, $|x + 2| < 5$. If $\varepsilon > 0$ is given, we can choose $|x - 2| < \varepsilon/5$.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta = \min 1, \varepsilon/5$. Then if $0 < |x-2| < \delta$, we have $1 < x < 3$ and $|x+2| < 5$. This gives:

$$\begin{aligned} |x^2 - 4| &= |x-2||x+2| \\ &< 5|x-2| \\ &< 5 \cdot \frac{\varepsilon}{5} \\ &= \varepsilon \end{aligned}$$

By the definition of the limit, we have $\lim_{x\to 2} x^2 = 4$. □

Example: Using the definition of the limit, prove that:

$$\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$$

Once again, we start at the end:

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{x-2}{2x}\right|$$

Since $|x-2|$ can be controlled arbitrarily, impose the condition $|x-2| < 1$. Then $1 < x < 3$. We see that $1 < |x|$. Then:

$$\frac{1}{|x|} < 1$$

If $|x-2| < 1$, we see that:

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{|x-2|}{2}$$

Therefore, we should choose $\delta = \min\{1, 2\varepsilon\}$.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta = \min\{1, 2\varepsilon\}$. Then if $0 < |x-2| < \delta$, we have that $1 < x < 3$ and $1 < |x|$. Then:

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{2}\right| &= \left|\frac{x-2}{2x}\right| \\ &< \frac{|x-2|}{2} \\ &< 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, by the definition of the limit, we have $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$. □

Example: Using the definition of the limit, prove that:

$$\lim_{x\to 9} \sqrt{x} = 3$$

Once again, we start at the end:

$$|\sqrt{x} - 3| = \frac{|x-9|}{\sqrt{x}+3} < \frac{|x-9|}{3}$$

If we choose $\delta = 3\varepsilon$, we are done.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta = 3\varepsilon$.

$$\begin{aligned} |\sqrt{x} - 3| &= \frac{|x-9|}{\sqrt{x}+3} \\ &< \frac{|x-9|}{3} \\ &< 3 \cdot \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

This completes the proof. □

9.2. **Properties of Limits.**

Theorem 1. *(from text) Suppose that $f(x)$ is a function defined on an open interval I around a, except possibly at a itself.*

1. *Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $x_n \to a$ in $I\backslash\{a\}$, we have $f(x_n) \to L$.*
2. *Suppose there are two sequences $x_n \to a$ and $y_n \to a$ in $I\backslash\{a\}$ such that $f(x_n) \to L_1$ and $f(y_n) \to L_2$. If $L_1 \neq L_2$, then $\lim_{x\to a} f(x)$ does not exist.*

Proof. Suppose that $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$ be given. Suppose $x_n \to a$. There is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. There is an $N > 0$ such that if $n \geq N$, then $|x_n - a| < \delta$. In this case, we have $|f(x_n) - L| < \varepsilon$.

For the converse, suppose that there the limit is not L. Then there is an $\varepsilon > 0$ such that for all $\delta > 0$, there is an x satisfying $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon$. For $\delta = 1/n$, let x_n be such a point. Then $x_n \to a$, but $f(x_n)$ does not converge to L. This contradicts the hypothesis.

The second statement follows easily from the first. □

Theorem 2. *Suppose that f and g are defined in an open interval I, except possibly at a point a. Suppose also that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then the following are hold:*

1. *$\lim_{x\to a} c \cdot f(x) = c \cdot L$, where c is any real number.*
2. *$\lim_{x\to a} f(x) \pm g(x) = L \pm M$*
3. *$\lim_{x\to a} f(x)g(x) = LM$*
4. *$\lim_{x\to a} f(x)/g(x) = L/M$, if $g(x) \neq 0$ in I and $M \neq 0$.*

Proof. These easily follow from the corresponding properties of sequences. For example, suppose that $x_n \to a$ in $I\backslash\{a\}$. Then $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. By the properties of sequences, the have:

$$\lim_{n\to\infty} f(x_n) + g(x_n) = L + M$$

Since the choice of sequence was arbitrary, the proof is complete.

We present a common alternate argument for the third property. Starting at the end:

$$|f(x)g(x) - LM| = |f(x)g(x) - Lg(x) + Lg(x) - LM| \leq |g(x)||f(x) - L| + |L||g(x) - M|$$

To make the estimate hold, we need to control the size of each term in the sum. First, control the size of $|g(x)|$. Choose $\varepsilon_1 > 0$ such that $M + \varepsilon_1 \neq 0$ and $M - \varepsilon_1 \neq 0$. There is a $\delta_1 > 0$ such that if $|x-a| < \delta_1$, then $M-\varepsilon_1 < g(x) < M+\varepsilon_1$. Let $K = \max\{|M-\varepsilon_1|, |M+\varepsilon_1|\}$. Then $|g(x)| < K$.

Choose $\delta_2 > 0$ so that $|f(x) - L| < \frac{\varepsilon}{2K}$. Choose $\delta_3 > 0$ so that $|g(x) - L| < \frac{\varepsilon}{2(|L|+1)}$. Define $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, we have:

$$|f(x)g(x) - LM| \leq K \cdot \frac{\varepsilon}{2K} + |L| \cdot \frac{\varepsilon}{2(|L|+1)} < \varepsilon$$

□

Theorem 3. *Suppose that $f(x)$ is a function defined on an open interval I around a, except possibly at a itself. If $m \leq f(x) \leq M$ for all $x \in I\backslash\{a\}$ and $\lim_{x\to a} f(x) = L$, then $m \leq L \leq M$.*

Theorem 4 (Squeeze Theorem)**.** *Suppose that f, g, h are a functions defined on an open interval I around a, except possibly at a itself. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in I\backslash\{a\}$. and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.*

Proof. We know it is true for sequences. Apply that theorem to a sequence $x_n \to a$ and the sequences $f(x_n) \to L$ and $h(x_n) \to L$. □

9.3. **Other kinds of limits.** There are other kinds of limits which are commonly used in calculus: left-hand limits, right-hand limits, limits at infinity and infinite limits. In this section, we record these definitions and look at some examples.

Definition: Let I be a finite open interval.

(1) (Left-hand limit) Let $a \in I$ or a is the right-hand endpoint of I. Let f be a function defined on I except possible at a. Then we write $\lim\limits_{x\to a^-} f(x) = L$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < a - x < \delta$, then $|f(x) - L| < \varepsilon$.
(2) (Right-hand limit) Let $a \in I$ or a is the left-hand endpoint of I. Let f be a function defined on I except possible at a. Then we write $\lim\limits_{x\to a^+} f(x) = L$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Example: Compute the left and right hand limits of the following function at 0. What can be said about the limit at 0?

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Theorem 5. *Let I be an open interval and let $a \in I$. Let f be defined on I except possibly at I. Then $\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$.*

Proof. The class proved this in class. □

Definition

(1) (Limits at Infinity) Let $I = (a, \infty)$ and suppose $f(x)$ is defined on I. We write $\lim\limits_{x\to\infty} f(x) = L$ if for all $\varepsilon > 0$, there is a $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$. We write $\lim\limits_{x\to-\infty} f(x) = L$ if for all $\varepsilon > 0$, there is a $M < 0$ such that if $x < M$, then $|f(x) - L| < \varepsilon$.
(2) (Infinite Limits) Let I be an open interval and suppose that $f(x)$ is defined at every point of I except possibly at a. We write $\lim_{x\to a} f(x) = \infty$ if fall $M > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$. There are similar definitions for limits to $-\infty$ and left-hand and right-hand infinite limits.

Example: Use the definition of the appropriate limit to prove the following.

$$\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$$

$$\lim_{x\to \infty} \frac{1}{\sqrt{x}} = 0$$

$$\lim_{x\to \infty} \sqrt{x+1} - \sqrt{x} = 0$$

For the last one, you should use the squeeze theorem and the computation of the previous limit.

LECTURE 10: CONTINUITY

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

10.1. **Definition and Examples.** The definition of continuity is very similar to the definition of the limit. In this section we will introduce the $\varepsilon - \delta$ definition of continuity and show that several functions are continuous on their domains using this definition. Recall that one of the major goals of this course is for you to learn how to make $\varepsilon - \delta$ arguments.

Definition: Let I be an interval and $f : I \to \mathbb{R}$ a function. Let $c \in I$. We say that f is continuous at c if for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - c| < \delta$ (and $x \in I$) then $|f(x) - f(c)| < \varepsilon$. If f is continuous at every $c \in I$, then f is said to be continuous on I.

Example: Use the definition to prove that $f(x) = x^2 + 2x - 1$ is continuous on $\mathbb{R}$.

Start at the end. Let $\varepsilon > 0$ be given and let $x_0 \in \mathbb{R}$.

$$\begin{aligned} |x^2 + 2x - 1 - (x_0^2 + 2x_0 - 1)| &= |x^2 - x_0^2 + 2(x - x_0)| \\ &= \underbrace{|x - x_0|}_{\text{this we can control}} \cdot |x + x_0 + 2| \end{aligned}$$

So we need to control the size of $|x + x_0|$. Choose $\delta < 1$. Then we have:

$$2x_0 - 1 < x + x_0 < 2x_0 + 1$$

Note that both $|2x_0 - 1|$ and $|2x_0 + 1|$ are not zero. Therefore, $0 < \max\{|2x_0 - 1|, |2x_0 + 1|\}$. Call this number M. Choose $\delta = \min\{1, \frac{\varepsilon}{2M}, \frac{\varepsilon}{4}\}$. Then if $|x - x_0| < \delta.$, we have:

$$|f(x) - f(x_0)| \leq |x - x_0||x + x_0| + 2|x - x_0| < \frac{\varepsilon}{2M} \cdot M + 2 \cdot \frac{\varepsilon}{4} = \varepsilon$$

After writing it up pretty, this would be a proof.

Example: Use the definition to prove that $g(x) = 1/(x-1)^2$ is continuous on $(-\infty, 1)$.

Start at the end. Let $\varepsilon > 0$ be given and let $x_0 \in (-\infty, 1)$.

$$\begin{aligned} \left| \frac{1}{(x-1)^2} - \frac{1}{(x_0-1)^2} \right| &= \frac{|x_0^2 - 2x_0 + 1 - x^2 + 2x - 1|}{(x-1)^2(x_0-1)^2} \\ &\leq |x - x_0| \, \frac{|x_0 - 1| + |x - 1|}{(x-1)^2(x_0-1)^2} \\ &= |x - x_0| \left(\frac{1}{(x-1)^2|x_0 - 1|} + \frac{1}{|x-1|(x_0-1)^2} \right) \end{aligned}$$

Let $m = |x_0 - 1|/2$ and choose $0 < \delta < m$. Then $m < |x - 1| < 3m$.

$$\begin{aligned} \left| \frac{1}{(x-1)^2} - \frac{1}{(x_0-1)^2} \right| &\leq |x - x_0| \left(\frac{1}{(x-1)^2|x_0 - 1|} + \frac{1}{|x-1|(x_0-1)^2} \right) \\ &\leq |x - x_0| \left(\frac{1}{m^2|x_0 - 1|} + \frac{1}{m(x_0-1)^2} \right) \end{aligned}$$

In the last line, the right multiplicand is a real number depending only on x_0. We call this number A. Choose $\delta = \min\{m, \frac{\varepsilon}{A}\}$. If $|x-a| < \delta$, then:

$$\Delta f \leq |x-a| \cdot A < \frac{\varepsilon}{A} \cdot A = \varepsilon$$

After writing it up pretty, this would be a proof.

Example: Use the definition to prove that $h(x) = \sin(x)$ is continuous on $\mathbb{R}$.

We follow the proof which is outlined in the exercises. The goal is to show that for any $a, b \in \mathbb{R}$,

$$|\sin(a) - \sin(b)| \leq |a-b|$$

First we show that $\sin(x) \leq x$ for all $x \in [0, \pi/2]$. Recall that x is measured in radians and corresponds to the arclength of the arc subtended by the angle. Suppose that the angle intersects the unit circle at points $A = (1,0)$ and B. The length of the line segment $\overline{AB}$ is less than the arclength x. The right triangle with hypotenuse $\overline{AB}$ and vertical leg of height $\sin(x)$ shows that $\sin(x) \leq x$.

We now show that $|\sin(x)| \leq |x|$. If $|x| \geq 1$, then the result is trivially true. If $0 < x < 1$, then $0 < x < \pi/2$. Since $\sin(x)$ is an odd function, it is clear that $|\sin(x)| \leq |x|$ for all x.

For any real x and y, note that we have the following identities:

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \sin(y)\cos(x) \\ \sin(x-y) &= \sin(x)\cos(y) - \sin(y)\cos(x) \\ \sin(x+y) - \sin(x-y) &= 2\sin(y)\cos(x) \end{aligned}$$

If $x = (a+b)/2$ and $y = (a-b)/2$, then we may have $x + y = a$ and $x - y = b$. In light of our identities this gives:

$$|\sin(a) - \sin(b)| = \left|2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)\right| \leq 2\frac{|a-b|}{2}$$

Let $\varepsilon > 0$ be given and let $x_0 \in \mathbb{R}$. Choose $\delta = \varepsilon$. If $|x - x_0| < \delta$, then:

$$|\sin(x) - \sin(x_0)| \leq |x - x_0| < \varepsilon$$

After writing this up pretty, this would be a proof.

10.2. **Theorems about Continuity.**

Theorem 1. *(from text) Let I be an interval, $f : I \to \mathbb{R}$ and let $c \in I$. The following are equivalent (TFAE):*

1. *The function f is continuous at c.*
2. *The function f has a limit at c and $\lim_{x \to c} f(x) = f(c)$. Note: If c is an endpoint, this should be altered so that the appropriate left/right limit applies.*
3. *The sequence $\{f(x_n)\}$ converges to $f(c)$ for each sequence $x_n \to c$ in I.*

Proof. We prove (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Assume (1). Let $\epsilon > 0$ be given. By (1), there is a $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - f(c)| < \varepsilon$. It is certainly true for x with $0 < |x-a| < \delta$. So the limit exists and is equal to $f(c)$. The converse follows similarly.

Suppose (2). Suppose $x_n \to c$ in $I \backslash \{c\}$. In this case, we know by a previous theorem that $f(x_n) \to f(c)$. Suppose that $x_n \to c$ in I. If x_n is eventually constant, the theorem is trivial. Otherwise, consider the set $A = \{n : x_n \neq c\}$. There is a subsequence $x'_n \to c$ defined by A. This implies $f(x'_n) \to f(c)$. The complementary sequence trivially converges to $f(c)$.

If (3) holds, then we have that $\lim_{x\to c} f(x) = f(c)$ for every sequence $\{x_n\}$ in $I\backslash\{c\}$. □

Theorem 2. *Let I be an interval, $c \in I$, f and g functions defined on I which are continuous at c and let k be a constant. Then the functions $f+g$,$f-g$, kf, and fg are continuous at c. If $g(c) \neq 0$, then f/g is continuous at c.*

Proof. This is easy. Use the previous theorem. □

Corollary 3. *Let f and g be continuous functions on an interval I and let k be a constant. Then the functions $f+g$,$f-g$, kf, and fg are continuous on I. If $g(x) \neq 0$ for all x in I, then f/g is continuous on I.*

Proof. This is immediate. □

Theorem 4. *Let I be an interval, let $g : I \to \mathbb{R}$, let $c \in I$ and let f be a function which is defined on an interval J that contains $g(I)$. If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c. Hence if g is continuous on I and f is continuous on J, then $f \circ g$ is continuous on I.*

Proof. Suppose that $x_n \to c$ in I. Then $g(x_n) \to g(c)$ in $g(I) \subset J$. Then $f(g(x_n)) \to f(g(c))$. Hence, for ever sequence $x_n \to c$ in I, we have $(f \circ g)(x_n) \to (f \circ g)(c)$. □

Theorem 5.

(1) *A polynomial is continuous at every real number.*
(2) *A rational function is continuous at every real number for which it is defined.*
(3) *The six trigonometric functions are continuous at every real number for which they are defined.*

Proof. This is also very easy. The first follows from induction. The last follows from the fact that $\sin(x)$ and $\cos(x) = \sin(x + \pi/2)$. □

10.3. **Ways in which a function can fail to be continuous.** There are three implications that follow from the continuity of the function f at a point c

- $f(x)$ is defined at c.
- $\lim_{x\to c} f(x)$ exists.
- $\lim_{x\to c} f(x) = f(c)$

Examples: Determine why the functions below fail to be continuous.

$$f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$h(x) = \begin{cases} x - 1 & x \neq 0 \\ \sqrt{2} & x = 0 \end{cases}$$

$$i(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Definition: Let I be an interval, $f : I \to \mathbb{R}$ a function, and let $c \in I$.

- We say that f has a jump discontinuity at c if the left and right hand limits exits at c but the limit does not exist at c (i.e. the left and right hand limits do not agree).
- We say that f has a removable discontinuity at c if the limit exits at c but is not equal to $f(c)$.

LECTURE 11: INTERMEDIATE AND EXTREME VALUES

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

We will not cover most of the section in the book. We are interested most in the two main theorems and how to apply them. We will prove each of them and then discuss how to use the theorems to prove things.

11.1. Intermediate Value Theorem.

Theorem 1. *Let $f(x)$ be a continuous function on $[a,b]$. Suppose that N is a number between $f(a)$ and $f(b)$. Then there is a c in $[a,b]$ such that $f(c) = N$*

Proof. The proof relies on the Completeness Axiom. Suppose that $f(a) < N < f(b)$. Let $S = \{x \in [a,b] : f(x) < N\}$. This set is a bounded set of real numbers: it is bounded below by a and above by b. By the Completeness Axiom, this set has a supremum c. It is clear that $f(c) \leq N$. We must show that $f(c) \geq N$. Following the text, we choose a sequence of numbers which converges to c not in S. Let $c_n = c + (b-c)/n$. Note that c_n is in $[a,b]$ but $c_n \notin S$ (as $c_n > c$ for all n). Hence, $f(c_n) \geq N$. Now, $c_n \to c$, so the continuity of f implies that $f(c_n) \to f(c)$. This means that $f(c) \geq N$. Hence, $f(c) = N$. This completes the proof. □

Example: Show that the polynomial $f(x) = x^3 - 4x + 1$ has a root in the interval $[1,2]$. $f(1) = -2$. $f(2) = 1$. We know $-2 < 0 < 1$, so there is a c in $[a,b]$ such that $f(c) = 0$.

Corollary 2. *Let f be a continuous function on an interval I. Suppose that for some $r, s \in I$, $r < s$, that $f(r) = f(s) = 0$. Moreover, suppose that if $r < x < s$, then $f(x) \neq 0$. Then either $f(x) > 0$ for all $x \in (r,s)$ or $f(x) < 0$ for all $x \in (r,s)$*

Proof. Suppose that $f(a) < 0$ and $f(b) > 0$ for some $a, b \in (r,s)$. Then IVT tells you that there is a c in (r,s) such that $f(c) = 0$. This is a contradiction. □

Example: Show that the equation $\sin(x) = x^2$ has a positive real number solution.

Consider $f(x) = \sin(x) - x^2$. This is a continuous function on $\mathbb{R}$. Note that 1 radian is an angle strictly between 0 and $\pi/2$. Then $f(1) < 0$. We will show that $f(\pi/4) > 0$. We must compare $\sqrt{2}/2$ and $\pi^2/16$. This inequality holds as long as $\pi^2 < 8\sqrt{2}$. Note that $11 < 8\sqrt{2} < 12$. Also note that $3 < \pi < 2^5/10$. Then:

$$9 < \pi^2 < 2^{10}/100 < 11 < 8\sqrt{2} < 12$$

Thus $f(\pi/4) > 0$. By IVT, there is a number c on $[\pi/4, 1]$ such that $f(c) = 0$.

Example: Show that $f(x) = x^5 - 4x + 2$ has exactly 3 real roots. By IVT, there are roots in the intervals $(-2,-1)$, $(1/2, 3/4)$ and $(1,2)$. This means that there are at least three distinct real root. To show that that there are exactly 3, we will show the following:

- $f(x) < 0$ for $x \leq -2$. This is easy.
- $f(x) > 0$ for $x > 2$. This is also easy.
- $f(x) > 0$ on $[-1,0]$. On this interval, $-4 \leq (x^4 - 4) \leq -3$. Then $-3x \leq x(x^4 - 4) \leq -4x$. On this interval that line $-3x \geq 0$ and the line $-4x \leq 4$. Then $0 < 2 \leq f(x) \leq 6$.
- $f(x) > 0$ on $(0, 1/2)$. This argument is basically the same. $-4 < (x^4 - 4) < (1/16) - 4$ and $-4x < x(x^4 - 4) < (1/16 - 4)x$. This gives you that $-2 < x(x^4 - 4) < 1/32 - 2$. Hence, $0 < f(x) < 1/32$ on this interval.
- $f(x) < 0$ on $[3/4, 1]$. This argument is also the same. You can show that $f(x) < (-1/4)$.

- $f(x)$ is increasing on $[-2,-1]$ and $[1,2]$. It is enough to show that x^5-4x is increasing on these intervals. Suppose $x<y$. We need $x^5-4x \le y^5-4y$ or:
$$4 < \frac{y^5-x^5}{y-x} = y^4+x^3y+y^2x^2+y^3x+x^4$$
Note that on either interval, the sum is always positive. In fact, each summand is positive and at least one. Therefore, the sum is at least 5. Since the inequality holds, $f(x)$ is increasing on this interval.
- $f(x)$ is decreasing on $[1/2, 3/4]$. The argument is the same. If $x<y$, we need to show that:
$$y^4+x^3y+y^2x^2+y^3x+x^4 = \frac{y^5-x^5}{y-x} < 4$$
The sum on the right is at most $5(3/4)^4 < 2 < 4$. Thus, $f(x)$ is decreasing on this interval.

This shows that $f(x)$ has exactly one root in each of the intervals we used for IVT and no roots outside the union of the intervals. Hence, we have exactly three roots.

Theorem 3. *Every polynomial of odd degree has at least one real root.*

Proof. Suppose $f(x)$ has degree n and $f(x) = ax^n +$ (lower power terms). We first show that $f(x)$ has the same limits at infinity as does ax^n.
$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{ax^n} &= \lim_{x\to\pm\infty} 1 + (\text{terms of the form } \frac{C}{x^k},\ k\ge 1)\\ &= 1\end{aligned}$$
Similarly, one may show that $\lim_{x\to\infty} \frac{1}{f(x)} = 0$. This means that $\lim_{x\to\infty} f(x) = \pm\infty$. Since $f(x)/ax^n \to 1$, $f(x)$ and ax^n have the same sign for x sufficiently large. This implies that $f(x)$ and ax^n diverge to the same infinity.

When n is odd, the limit of ax^n as $x\to\infty$ is opposite of the the limit as $x\to-\infty$. Hence, there is some a, b for which $f(a)<0$ and $f(b)>0$. By IVT, there is a c such that $f(c)=0$. This completes the proof. □

Example: Let f be a continuous function on $[0,2]$ and suppose that $f(0)=f(2)$. Prove that there exists a $c\in[0,1]$ such that $f(c)=f(c+1)$.

Proof. Let $g(x) = f(x+1)-f(x)$. This function is continuous since it is formed using compositions and differences. We see that $g(0)=f(1)-f(0)$ and $g(1)=f(2)-f(1)=f(0)-f(1)$. Thus, $g(0)$ and $g(1)$ have opposite sign and we conclude by the IVT that there is c such that $g(c)=0$. From this we see that $f(c+1)=f(c)$. □

Theorem 4. *Let I be an interval and $f: I\to\mathbb{R}$ be a monotone function. If $f(I)$ is an interval, then f is continuous on I.*

Proof. We won't prove this. The proof is in the book. We need this to prove the next result. □

Example: Let $f:[a,b]\to\mathbb{R}$ be continuous and define $g:[a,b]\to\mathbb{R}$ by:
$$g(x) = \sup\{f(t) : a\le t\le x\}$$
Show that $g(x)$ is increasing and continuous.

It is clear that $g(x)$ is increasing. To show that it is continuous, it need only be shown that $g([a,b])$ is an interval. Suppose that $g(x)<z<g(y)$. We must show that $g(c)=z$ for some $c\in[a,b]$. On $[a,x]$, $f(t)$ achieves its maximum value at $f(c_1)$. In fact, this maximum value is $g(x)=f(c_1)$. Similarly, $g(y)=f(c_2)$ for some c_2. Now, $c_1<c_2$ by hypothesis. By the intermediate value theorem, there is a c, $c_1<c<c_2$ such that $f(c)=z$.

Let $S = \{t \in [a,b] : f(t) = z\}$. This set is nonempty and bounded below by c_1. Let $\gamma = \inf(S)$. Then $\gamma \geq c_1$. Since f is continuous, $f(\gamma) = z$(take a sequence in S which converges to the infimum). By construction, $f(t) < f(\gamma)$ for $a \leq t < \gamma$. Hence, $g(\gamma) = z$ and we are done.

Example: Suppose that $f : I \to \mathbb{R}$ is a continuous function and that $|f|$ is a constant. Prove that f is constant.

Proof. Suppose $|f(x)| = C$ for all $x \in I$. If $C = 0$, then clearly $f(x) = 0$ for all $x \in I$. If $C \neq 0$, then for every x, either $f(x) = C$ or $f(x) = -C$. Suppose f is not constant. Then there is an x such that $f(x) = C$ and a y such that $f(y) = -C$. Assume $x < y$. By IVT, there is a $z \in [x,y]$ such that $f(z) = 0$. This is a contradiction, so we must have that f is a constant. □

Example: Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that the range of f is a countable set. What can you say about the range of f?

Let $x, y \in [a,b]$, $x < y$. Suppose that $f(x) \neq f(y)$. There are uncountably many different values between $f(x)$ and $f(y)$ that are all achieved by f, since f is continuous. This is a contradiction, so we must have $f(x) = f(y)$ for all $x, y \in [a,b]$ i.e., it is a constant.

11.2. **Extreme Value Theorem.**

Theorem 5. *Let f be a continuous function defined on a closed interval $[a,b]$. There are numbers $c, d \in [a,b]$ such that $f(c) \leq f(x)$ for all $x \in [a,b]$ and $f(x) \leq f(d)$ for all $x \in [a,b]$. In other words, $f(x)$ achieves its maximum and minimum values.*

Proof. The content of this theorem concerns bounded sets. We need to show that the image of f is bounded and that it achieves it maximum and minimum value. The proof is thus divided into two parts:

(1) f is bounded on $[a,b]$.
(2) f achieves its max and min on $[a,b]$.

Suppose that f is not bounded. Then for all n there is an a_n such that $|f(a_n)| > n$. This sequence $\{a_n\}$ is bounded above by b and below by a. This sequence has a convergent subsequence $\{a_{n_q}\}$. However, the sequence $\{f(a_{n_q})\}$ does not converge. This contradicts the fact that f is continuous on $[a,b]$. Thus, f is bounded and the image of f is bounded.

A bounded set of real numbers has a supremum β and an infimum α. We will show that $f(c) = \alpha$ for some $c \in [a,b]$. For every natural number n, there is an x_n such that:

$$\alpha \leq f(x_n) < \alpha + \frac{1}{n}$$

The sequence $\{x_n\}$ is bounded, so it has a convergent subsequence $\{x_{n_q}\}$, $x_{n_q} \to c$. Since $f(x)$ is continuous, we have:

$$\lim_{n\to\infty} f(x_{n_q}) = f(c)$$

By construction, $f(x_n) \to \alpha$. Thus, $f(c) = \alpha$. The book covers the case showing that β is achieved. □

Theorem 6. *Let I be an interval and let $f : I \to \mathbb{R}$.*

(1) *If f is continuous on I and not a constant function, then $f(I)$ is an interval.*
(2) *If f is continuous on I and not a constant function on I, and I is closed and bounded, then $f(I)$ is closed and bounded on I.*

Proof. Let $f(x) < z < f(y)$. By IVT, Then there is a c such that $f(c) = z$. Then $z = f(c) \in f(I)$ and we see that $f(I)$ is an interval.

If I is closed and bounded, $I = [a, b]$ for some $a, b \in \mathbb{R}$. By EVT, there is a $c, d \in [a, b]$ such that $f(c) \leq f(x)$ and $f(d) \geq f(x)$. The image of $[a, b]$ is the interval $[f(c), f(d)]$. □

In class, I gave some examples to show that all conditions of the EVT are necessary.

LECTURE 12: UNIFORM CONTINUITY

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

Once again, we will not cover all of the topics in this section. We focus on several motivating examples, the definition of uniform continuity, and a very important theorem about continuous functions.

12.1. **Motivating Examples.**

(1) In the lecture on continuity, we looked at the example $f(x) = x^2 + 2x - 1$. Given an $\varepsilon > 0$, we found that if we set $\delta = \min\{1, \varepsilon/4, \varepsilon/2M\}$, where $M = \max\{|2x_0 - 1|, |2x_0 + 1|\}$. In this case, we see that the value of δ we found depended on the point x_0.

(2) Consider any line $y = mx + b$, $m \neq 0$. If $\varepsilon > 0$ is given, then we may choose $\delta = \varepsilon/|m|$. If $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. In this case, the δ may be chosen so that it does not depend on a point $x_0 \in \mathbb{R}$.

(3) Consider the example $f(x) = \sin(x)$. We have the identity:

$$|\sin(x) - \sin(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}$. If we choose $\delta = \varepsilon$, we see that we may choose δ so that it does not depend on a point x_0.

12.2. **Uniform Continuity.** In the examples we have seen, sometimes the δ depends on the value of x_0 and the value of ε. If the same δ works for every x_0 in an interval I, we say that f is uniformly continuous on I.

Definition: Let $f(x)$ be a function defined on an interval I. We say that f is uniformly continuous on I if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in I$ satisfying $|x - y| < \delta$.

Theorem 1. *If f is uniformly continuous on I, then f is continuous on I.*

Proof. Let $c \in I$ and $\varepsilon > 0$ be given. There is a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Thus, f is continuous at c. □

Example:(from text) $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Let $\varepsilon = 2$, $\delta > 0$ be given. Set $x = 1/\delta$, $y = \delta + 1/\delta$.

12.3. **An Important Theorem.** In this section, we prove an important theorem about continuous functions.

Definitions: Let $S \subset \mathbb{R}$. An *open covering (by intervals)* of S is a collection of open intervals $\{I_n\}$ so that $S \subset \cup_n I_n$. A subcovering of an open covering $A = \{I_n\}$ is a subset of A, $\{I_{n_k}\}$, such that $S \subset \cup_k I_{n_k}$.

Example: Let $S = [0,1]$. Let $I_1 = (-1, 1/10)$. For $n \geq 2$, let $I_n = (1/n, 3/2)$. Then $\{I_n\}$ is an open covering (by intervals) of S. The set:

$$\{I_1, I_{11}\}$$

is a subcovering with finitely many elements.

The *open sets* of $\mathbb{R}$ are those sets which can be written as a union of open intervals. The union may be over an infinite number of intervals or even over no intervals. Both $\emptyset$ and $\mathbb{R}$ are open sets. This defines the *topology* of $\mathbb{R}$. It is the same as the topology defined by the metric $d(x,y) = |x-y|$.

Given any topological space X, and a set $A \subset X$, an open covering of A is a set of open sets $\{S_i : i \in I\}$ such that $A \subset \bigcup_{i \in I} S$. We say that A is *compact* if every open covering has a finite subcovering.

Thus, this section is really about the topology of the real numbers. To avoid some of the generalities necessary to understand compact sets, we will instead prove the following remarkable theorem. It does everything that we need to prove theorems about uniform continuity.

Theorem 2 (The Lebesgue Number Lemma)**.** *Let $A = \{I_n\}$ be an open covering (by intervals) of $[a,b]$. There is a $\delta > 0$(called the Lebesgue number) such that if $[x,y] \subset [a,b]$ and the length of $[x,y]$, l([x,y])=y-x, is less than δ, then $[x,y] \subset I_k$ for some k.*

Proof. Suppose otherwise. Suppose that for every $\delta > 0$, there is an interval $J_\delta \subset [a,b]$ such that the length of J_δ is less than δ, but $J_\delta \not\subset I_n$ for any n.

Consider the intervals $J_{1/n}$ where n is a positive integer. Choose $x_n \in J_{1/n}$. The sequence $\{x_n\}$ is bounded. Hence it has a convergent subsequence $\{x_{n_k}\}$. Suppose $x_{n_k} \to x$.

Since $x \in [a,b]$, there is an I_j such that $x \in I_j$. Choose $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subset I_j$.
Choose k so that $|x_{n_k} - x| < \varepsilon/2$ and $1/n_k < \varepsilon/2$. Note that if $r, s \in J_{1/n_k}$, then $|r-s| < \varepsilon/2$. We wish to show that if $u \in J_{1/n_k}$, then $u \in I_j$:

$$\begin{aligned} |u-x| &= |u - x_{n_k} + x_{n_k} - x| \\ &\leq |u - x_{n_k}| + |x_{n_k} - x| \\ &< \varepsilon \end{aligned}$$

This means that $J_{1/n_k} \subset (x-\varepsilon, x+\varepsilon) \subset I_j$. This is a contradiction. Hence, the result is true. □

Theorem 3. *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then it is uniformly continuous on $[a,b]$.*

Proof. The proof given here is very different than the one given in your text.

Let $c \in [a,b]$ and $\varepsilon > 0$ be given. Since f is continuous at c, there is a $\delta_c > 0$ such that if $x \in [a,b]$ and $|x-c| < \delta_c$, then $|f(x) - f(c)| < \varepsilon/2$. Let $I_c = (c - \delta_c, c + \delta_c)$.

The set $\{I_c\}$ is an open covering of $[a,b]$. By the Lebesgue Number Lemma, there is a $\delta > 0$ such that if $|x-y| < \delta$, then $x, y \in I_{c_k}$ for some k. This implies:

$$|f(x) - f(y)| = |f(x) - f(c_k) + f(c_k) - f(y)| \leq |f(x) - f(c_k)| + |f(y) - f(c_k)| < \varepsilon$$

Hence, f is uniformly continuous. □

12.4. **Some problems and their solutions.**

Theorem 4. *If $A = \{I_n\}$ is a open covering of the interval $[a,b]$, then A has a finite subcovering.*

Proof. For this, we follow the proof of Theorem 8.15 from you text. Let E be the set of elements x of $(a, b]$ such that $[a, x]$ has a finite subcover by A. The aim is to show that the supremum of E is b. To do this, we must show the following:

- E is nonempty.
- E is bounded above.
- E contains its supremum.
- The supremum of E is b.

For the first bullet, take an interval I_n such that $a \in I_n$. By the above lemma, there is an $x \in I_n$, $x \neq a$, such that $[a, x] \subset I_n$. Since $[a, x]$ is covered by I_n, we have $x \in E$.

The set E is bounded above by b, by definition. By the Completeness Axiom, E has a supremum, $\beta = \sup(S)$.

Next we show that $\beta \in E$. Certainly, $\beta \in [a, b]$. There is a k such that $\beta \in I_k$. There is a $c \in [a, b]$ such that $[c, \beta] \in I_k$. Since $c < \beta$, it is not an upper bound of E. This means that there is an $x \in E$ such that $c < x \leq \beta$. The set $[a, x]$ is thus finitely covered by A. Also, $[x, \beta] \subset I_k$. Thus the set $[a, \beta] = [a, x] \cup [x, \beta]$ is finitely covered by A. Hence $\beta \in E$.

Finally we show that $\beta = b$. Suppose otherwise. Then $\beta < b$. Using I_k as defined above, we have an element $w \in I_k$ such that $[\beta, w] \subset I_k$ and $\beta < w < b$. The set $[a, w] = [a, \beta] \cup [\beta, w]$ is finitely covered by A, so $w \in E$. This contradicts the fact that $\beta = \sup(E)$. Thus $\beta = b$. $\square$

LECTURE 13: THE DERIVATIVE

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

13.1. **The Derivative.** We begin by defining the derivative and describing its properties geometrically.

Definition: Let $f(x)$ be defined in an interval around a. Then the derivative of f at a is given by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists. Alternatively, we may write:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The three most common interpretations of the derivative are:

(1) The derivative is the limit of the slopes of the secant lines.
(2) The derivative is the slope of the tangent line.
(3) The derivative is the instantaneous rate of change at a point.

In class we drew the standard picture illustrating the derivative.

Example: Compute the derivative of $f(x) = x^2$ at $x = -1$ using the defintion of the derivative.

Example: Compute the derivative of $f(x) = \sqrt{x}$ at $x = 9/4$ using the definition of the derivative.

Example:(The Power Rule) Show that if n is a postive integer and $f(x) = x^n$, then $f'(a) = na^{n-1}$. Let us first explicitly compute the case for $f(x) = x^3$.

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{x^3 - a^3}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} \\ &= 3a^2 \end{aligned}$$

For the general rule, we need the following general factorization technique.

$$(x-a)(x^n + ax^{n-1} + a^2x^{n-1} + \ldots + a^n) = x^{n+1} - a^{n+1}$$

Let $f(x) = x^n$, where n is a positive integer.

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-1} + a^2x^{n-1} + \ldots + a^{n-1})}{x - a} \\ &= na^{n-1} \end{aligned}$$

13.2. **Ways a function can fail to be differentiable:** A function is said to be differentiable at a if the derivative exists at a.

Theorem 1. *If f is differentiable at a, then f is continuous at a. Hence, if f is differentiable on an interval, then it is continuous on that interval.*

Proof. Such a function is certainly defined at a, so we only need to check the limit:

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)\\ &= f'(a)\cdot 0\\ &= 0\end{aligned}$$

Therefore, $\lim_{x\to a} f(x) = f(a)$ and we conclude that f is continuous at a. □

It is not true that a continuous function is necessarily differentiable. For example, consider $f(x) = |x|$ is not differentiable at 0. We show this by investigating the limit which defines the derivative.

(1) A function which is not continuous is not differentiable (this is exactly what the above theorem says).
(2) A function is not differentiable at a cusp or a corner. A cusp or a corner occurs in a continuous function when the left and right hand limits for the derivative exist, but are not equal to one another. An example is $f(x) = |x|$.

    Example: Graph the following function and show that it is not differentiable at every point.

$$g(x) = \begin{cases} 2x-2 & x<1 \\ x-1 & x\geq 1 \end{cases}$$

(3) A function which has a vertical tangent line at a point is not differentiable at that point.

    Example: Use the limit definition of the derivative to show that the following function is not differentiable at the origin because it has a vertical tangent line. Also, sketch the graph of the function.

$$f(x) = x^{1/3}$$

13.3. **Differentiation Rules.** In this section, we establish the well-known Differentiation Rules.

Theorem 2. *If f and g are differentiable at a and k is an constant, then:*

(1) *$k\cdot f$ is differentiable at a and $(kf)'(x) = kf'(x)$.*
(2) *$f\pm g$ is differentiable at a and $(f\pm g)' = f'\pm g'$*

The product rule is for differentiating a product of two functions. Be aware that:

$$\frac{d}{dx}f(x)\cdot g(x) \neq f'(x)\cdot g'(x)$$

We will find a correct formula for the product rule and prove that the formula is valid.

To derive the correct formula, think of the following picture:

	$g(x)$	Δg
$f(x)$		
Δf		

$$\Delta f = f(x+h) - f(x)$$
$$\Delta g = g(x+h) - g(x)$$

If we know the rate at which $f(x)$ and $g(x)$ are changing, how do you find the rate at which $f(x)\cdot g(x)$ is changing? The length of the vertical side is $f(x+h)$ The length of the horizontal side is $g(x+h)$. Then:

$$\Delta fg = f(x+h)g(x+h) - f(x)g(x)$$

In terms of the picture, this is the area in the three rectangular regions:

$$\Delta fg = g(x)\cdot\Delta f + f(x)\Delta g + \Delta f\cdot\Delta g$$

Suppose that $\Delta x = (x+h) - x$ is small. The derivative is approximated by:

$$\begin{aligned}\frac{d}{dx}f(x)g(x) &\approx \frac{\Delta fg}{\Delta x}\\ &= g(x)\cdot\frac{\Delta f}{\Delta x} + f(x)\frac{\Delta g}{\Delta x} + \frac{\Delta f\cdot\Delta g}{\Delta x\Delta x}\Delta x\\ &\approx f'(x)g(x) + f(x)g'(x) + f'(x)g'(x)\Delta x\end{aligned}$$

As $\Delta x\to 0$, the last term goes to zero. It turns out that the approximation gets better and better as $\Delta x\to 0$ (in fact, the error in the approximation goes to zero) and we have:

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$$

Theorem 3 (The Product Rule)**.** *Suppose that f and g are differentiable. Then $f\cdot g$ is differentiable and its derivative is given by:*

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$$

Proof. To prove the theorem, we need to use the limit definition of the derivative. The proof uses a trick which is known as the Add-and-Subtract trick.

$$\begin{aligned}(f\cdot g)'(x) &= \lim_{h\to 0}\frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)\cdot g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h}\end{aligned}$$

We will show later that if g is differentiable at a point, it is continuous at a point. Thus, under the hypotheses of the theorem, $\lim_{h\to 0} g(x+h) = g(x)$. It follows from the limit laws that:

$$\begin{aligned}(f\cdot g)'(x) &= g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= g(x)f'(x) + g'(x)\cdot f(x)\end{aligned}$$

□

Theorem 4 (The Quotient Rule)**.** *Suppose that f and g are differentiable on an interval I and g is nonzero on I. Then (f/g) is differentiable at every $x\in I$ and:*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Proof. Exercise. □

13.4. **The Chain Rule.** First let us recall the statement of the Chain Rule.

Theorem 5 (The Chain Rule)**.** *Suppose that f and g are functions so that the composition $f \circ g$ is defined for all x in some interval I (In particular, f is defined on some interval J containig $g(I)$). Suppose that g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a and:*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Equivalently, if $y = f(x)$ and $x = g(t)$, we have:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Example: Use the Chain Rule to find the following derivatives.

(1) $f(x) = (x^2 + 1)^{100}$
(2) $f(x) = \sin(x^2)$
(3) $f(x) = (3 - 2x)^{-2}$

We will look at three proofs of the Chain Rule. The first is a geometric proof. The second is an incorrect proof. The third is a completely solid proof.

Geometric Proof (Bonus): Consider two gears of radius r_1 and r_2 which are connected by a closed chain encircling both gears. Each gear turns on an axel which is independent of the other gear. One of the gears is being turned with an angular velocity of $d\theta_1/dt$. What is the angular velocity of the second gear? The second gear has angular velocity $d\theta_2/dt$. If the first gear moves a little bit, the chain moves a little bit. The arclength of the chain that moves is $s = r_1\Delta\theta_1$. The second gear is forced to move the same amount and there is a corresponding change in θ_2: $s = r_2\Delta\theta_2$. hence we have:

$$\frac{r_1}{r_2} = \frac{\Delta\theta_2}{\Delta\theta_1}$$

If we let $\Delta\theta_1 \to 0$, then we have:

$$\frac{r_1}{r_2} = \frac{d\theta_2}{d\theta_1}$$

In general, we se that as a function of time, $\theta_2(t) = \frac{r_1}{r_2} \cdot \theta_1(t)$. This gives:

$$\frac{d\theta_2}{dt} = \frac{r_1}{r_2}\frac{d\theta_1}{dt} = \frac{d\theta_2}{d\theta_1}\frac{d\theta_1}{dt}$$

Incorrect Proof: Compute the limit directly:

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{g(x) - g(a)}{g(x) - g(a)} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\
&= f'(g(a)) \cdot g'(a)
\end{aligned}$$

This is incorrect because it might be true that $g(x) - g(c) = 0$ for some x in every inerval around c. For example, $f(x) = x^2 \sin(1/x)$ has a 0 on every open interval containing 0.

Solid Proof: This is the very clever argument that you will find in your textbook and many other books. Define a function F as follows:

$$F(x) = \begin{cases} \frac{f(x)-f(g(a)}{x-g(a)} & x \neq g(a) \\ f'(g(a)) & x = g(a) \end{cases}$$

Since f is differentiable at $g(a)$, we have that $\lim_{x \to g(a)} F(x) = F(g(a))$ and F is continuous at $g(a)$. The following holds for all $y \in J$:

$$F(y)(y - g(a)) = f(y) - f(g(a))$$

Since $g(x) \in J$ for all $x \in I$, we have that:

$$F(g(x))(g(x) - g(a)) = f(g(x)) - f(g(a))$$

Now we compute the derivative:

$$\begin{aligned} (f \circ g)'(a) &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\ &= \lim_{x \to a} \frac{F(g(x))(g(x) - g(a))}{x - a} \\ &= \lim_{x \to a} F(g(x)) \frac{g(x) - g(a)}{x - a} \\ &= f'(g(a)) \cdot g'(a) \end{aligned}$$

This proof may lead one to despair. "How did someone come up with that???" Fear not. It basically just a slick way of writing an estimation argument. It would take us too far afield to do this. There is an acceptable version of this on the Wikipedia. Look up "Chain Rule".

13.5. **Derivatives of Trigonmetric Functions.** In this section, we will compute the derivatives of $\sin(x)$ and $\cos(x)$. Both derivatives rely on computing a special limit. The proof of this limit is so beautiful, that it can't be missed:

Theorem 6.

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Proof. It is possible to use l'Hôpital's rule for this. But there is a much more enlightening geometric proof. For the moment, suppose $x > 0$. In this case, note that x is measured in radians. In the picture, there are three areas. There is a triangle, a sector, and a larger triangle that contains them all. The lengths of the base of the large triangle is 1. The height, y, can be found using similar triangles:

$$\frac{y}{\sin(x)} = \frac{1}{\cos(x)}$$

The areas of the regions satisfy the following:

$$\frac{1}{2} \sin(x) \cos(x) \leq \pi \cdot \frac{x}{2\pi} \leq \frac{1}{2} \tan(x)$$

Since $\sin(x) > 0$ when $x > 0$, we can rearrange the quantity to obtain:

$$\cos(x) \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}$$

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}$$

We know that $\cos(x)$ is continuous on $\mathbb{R}$. As $x \to 0^+$, both the left and right side of the inequality converge to 1. Hence, we have:

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$$

There is a similar argument which gives you the left hand limit, or you can use the fact that $\sin(x)/x$ is an even function. □

Another important limit can be computed as follows:

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{x} &= \lim_{x\to 0}\frac{(1-\cos(x))(1+\cos(x))}{x(1+\cos(x))}\\ &= \lim_{x\to 0}\frac{\sin^2(x)}{x(1+\cos(x))}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\sin(x)\cdot\frac{1}{1+\cos(x)}\\ &= 0\end{aligned}$$

Now, we are ready to compute the derivative of $\sin(x)$

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\right)\\ &= \lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\cdot\frac{\sin(h)}{h}\right)\\ &= \cos(x)\end{aligned}$$

For $\cos(x)$, you can do something similar, or you can use the Chain Rule:

$$\begin{aligned}\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sin\left(x+\frac{\pi}{2}\right)\\ &= \cos\left(x+\frac{\pi}{2}\right)\cdot 1\\ &= \cos(x)\cos(\pi/2)-\sin(x)\sin(\pi/2)\\ &= -\sin(x)\end{aligned}$$

It is now very easy to find the derivatives of the other trigonometric functions. I will leave that as an exercise.

LECTURE 14: THE MEAN VALUE THEOREM AND ITS APPLICATIONS

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

14.1. **Critical points, Fermat's Theorem.**

Theorem 1 (Fermat's Theorem)**.** *If $f : I \to \mathbb{R}$ is differentiable at $a \in I$ and f has a relative extrema at a, then $f'(a) = 0$.*

Proof. Without loss of generality, we assume that f has a local maximum at a. We compute the derivative by looking at the left and right hand limits. On the left, the difference quotient is always > 0 near a. On the right, the difference quotient is always < 0 near a. Then since f is differentiable,

$$f'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \geq 0$$

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \leq 0$$

Therefore, $f'(a) = 0$. □

Important Examples: It is not true that if $f'(a) = 0$, then you have a relative extrema at a. For example, $f(x) = x^3$ does not have a local extrema at 0. The hypothesis that f is differentiable is necessary. The function $f(x) = |x|$ has a local minimum at 0, but $f'(0) \neq 0$ since it does not exist there.

Definition A critical point of a function $f(x)$ is a place where the derivative is 0 or does not exist.

Example Find all critical point of the function $f(x) = x^{2/3}(x-2)$.

14.2. **Rolle's Theorem, Mean Value Theorem.** Rolle's theorem is a special case of the Mean Value Theorem. It is generally proved first as a lemma which gives us MVT for free.

Theorem 2 (Rolle's Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a function. Also suppose that the following conditions are satisfied:*

1. *f is continuous on $[a, b]$*
2. *f is differentiable on (a, b)*
3. *$f(a) = f(b)$*

Then there is a $c \in (a, b)$ such that $f'(c) = 0$.

Proof. There are three cases to consider. If f is a constant on (a, b), then the result is trivial. First, suppose that there is a point $r \in (a, b)$ such that $f(r) > f(b)$. By the Extreme Value Theorem, there is a point c such that $f(c) \geq f(x)$ for all $x \in (a, b)$. This point is also a local extrema. Hence, by Fermat's theorem, $f'(c) = 0$.

Lastly, suppose there is a point $r \in (a, b)$ such that $f(r) < f(a)$. Using the EVT again, we find a point c such that $f'(c) = 0$. □

Example: Show that the function $f(x) = x^6 - 10x^3 + 4x - 3$ has at least one critical point in the interval $[-1, 3]$. Note that $f(-1) = 4$, $f(0) = -3$, $f(1) = -8$ and $f(3) = 468$. Then by IVT, f has a root in $(-1, 0)$ and $(1, 3)$. In particular, there are two points $a \in (-1, 0)$ and $b \in (1, 3)$ such that $f(a) = f(b) = 0$. Thus, by Rolle's Theorem, there is a point c between a and b such that $f'(c) = 0$. We don't know what that point is or if there are many such points, but we at least know that there is one.

Theorem 3 (The Mean Value Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a function satisfying the following hypotheses.*

(1) *f is continuous on $[a, b]$*
(2) *f is differentiable on (a, b)*

Then there is a $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. The proof of this theorem uses Rolle's Theorem. We take our function $f(x)$ and shift it down so that it satisfies the conditions of Rolle's Theorem. We then get the existence statement from Rolles and translate it back to the desired existence statement.

Consider the function $g : [a, b] \to \mathbb{R}$ defined below:

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

Note that $g(a) = g(b) = 0$. This satisfies the conditions of Rolle's Theorem. Then there is a $c \in (a, b)$ such that $g'(c) = 0$. Now, we can compute the derivative of g at c using our differentiation formulas.

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

This proves the theorem. □

Example: Show that the MVT guarantees that there is a point c such that $f'(c) = 12$. Find all points in the given interval which satisfy this equation.

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 7, \;\; x \in [-6, 6]$$

We have $f(-6) = -83$ and $f(6) = 61$. There exists a $c \in (a, b)$ such that $f'(c) = 144/12 = 12$. We have $f'(x) = x^2 - x$. The equation is satisfied when $x^2 - x - 12 = 0$. This occurs at $x = 4$ and $x = -3$.

Example: Is it possible that there exists a function $f : [0, 2] \to \mathbb{R}$ such that $f(0) = 1$, $f(2) = \pi$ and $f'(x) < 1$ for all $x \in (0, 2)$? The answer is NO. Apply the MVT.

Example: Let $f : [a, b] \to [a, b]$ be a continuous function that is differentiable on (a, b). Show that there is a point c such that $f(c) = c$. In other words, show that f has a *fixed point*. If $|f'(x)| < 1$ for all $x \in (a, b)$, show that there is at most one fixed point.

Proof. Consider the function $g(x) = f(x) - x$. We see that $g(a) \geq 0$ and $g(b) \leq 0$. If $g(a) = 0$ or $g(b) = 0$, then there is a fixed point. Otherwise, $g(a) > 0$ and $g(b) < 0$. Then by IVT, there is a c such that $0 = g(c) = f(c) - c$.

Suppose that $|f'(x)| < 1$ and that $f(r) = r$ and $f(s) = s$, $r < s$. By MVT, there is a $c \in (r, s)$ such that:

$$f'(c) = \frac{f(s) - f(r)}{s - r} = 1$$

This is a contradiction. Hence, this function can have at most one fixed point. □

14.3. **Increasing/Decreasing Test, First Derivative test, Second Derivative Test.**

Theorem 4 (Increasing/Decreasing Test)**.** *Let f be a real valued function on an interval $I = [a, b]$ that is differentiable on the interior of I and continuous on I.*

(1) *If f is increasing on I, then $f'(x) \geq 0$ for all x in I. If $f'(x) > 0 (f'(x) \geq 0)$ for all x in I, then $f(x)$ is strictly increasing (increasing) on I.*

(2) *If f is decreasing on I, then $f'(x) \leq 0$ for all x in I. If $f'(x) < 0 (f'(x) \leq 0)$ for all x in I, then $f(x)$ is strictly decreasing (decreasing) on I.*
(3) *If $f'(x) = 0$ for all x in I, then f is a constant.*

Proof. Suppose that f is increasing. Then if $x < y$, we have $f(x) \leq f(y)$. Compute the right hand limit defining the derivative:

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Since the difference quotient is always positive, we have $f'(a) \geq 0$.

Now suppose that $f'(x) \geq 0$ for all x. Let $x, y \in I$, $x < y$. We would like to show that $f(x) \leq f(y)$. Note that f is continuous on $[a, b]$ and differentiable on (a, b). Then there is a $c \in (x, y)$ such that:

$$0 \leq f'(c) = \frac{f(y) - f(x)}{y - x}$$

Therefore, $f(x) \leq f(y)$.

Suppose that $f'(a) = 0$ for all $a \in I$ and $x, y \in I$, x, y. Then there is a $c \in [x, y]$ such that $0 = f'(c) = (f(y) - f(x))/(y - x)$. This implies that $f(x) = f(y)$ and f is a constant function. □

Corollary 5. *Let f and g be two functions continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)$ for all $x \in (a, b)$. then $f - g$ is a constant on $[a, b]$*

Proof. It is easy to see that $f - g$ is a constant on (a, b). It is a constant on $[a, b]$ because it is continuous there. The one-sided limits at the endpoints are limits of constant functions. □

Theorem 6 (First Derivative Test)**.** *Let $f : (a, b) \to \mathbb{R}$ be continuous on (a, b) and differentiable on $(a, b) \backslash \{c\}$. Suppose that f has a critical point at c.*
(1) *If $f' > 0$ on (a, c) and $f' < 0$ on (c, b), then f has a local maximum at c.*
(2) *If $f' < 0$ on (a, c) and $f' > 0$ on (c, b), then f has a local minimum at c.*

Proof. We prove only the first statement. The second follows similarly. Suppose that $f(y) \geq f(c)$ for some $y \in (a, c)$. Then by the MVT applied to f on $[y, c]$, we have that there is a $z \in (y, c) \subset (a, c)$ such that $f'(z) = (f(c) - f(y))/(c - y) \leq 0$. This is a contradiction. Thus, $f(c) > f(y)$ for all $y \in (a, c)$. Similarly, $f(c) > f(y)$ for all y in (c, b). Thus, f has a local max at c. □

Theorem 7 (Second Derivative Test)**.** *Suppose that $f : (a, b) \to \mathbb{R}$ has a second derivative at every point in (a, b). Suppose that $c \in (a, b)$ is a critical point of f (here this means $f'(c) = 0$).*
(1) *If $f''(c) > 0$, then f has a local minimum at c.*
(2) *If $f''(c) < 0$, then f has a local maximum at c.*

Proof. We prove the first fact only as the second assertion follows similarly. Let $c_n \to c$ be an increasing sequence with $c_n < c$ and $f(c_n) \leq f(c)$ for all n. Then $f(c) - f(c_n) \geq 0$ for all n. By MVT, there exists a $d_n \in (c_n, c)$ such that $f'(d_n) \geq 0$. Note that $d_n \to c$. Since $f''(c)$ exists, we have:

$$f''(c) = \lim_{n \to \infty} \frac{f'(c) - f'(d_n)}{c - d_n} \leq 0$$

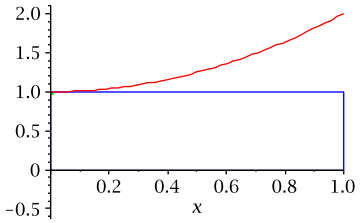
This is a contradiction. Thus, $f(c) < f(x)$ for all x in some interval (r, c). Similarly, $f(c) < f(x)$ for all x in some interval (c, s). We see that f has a local minimum at c. □

LECTURE 15: THE RIEMANN INTEGRAL

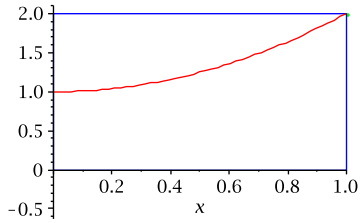
These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

15.1. **Area Below a Curve.** Let us recall the left-hand and right-hand endpoint methods for finding the area below a curve. Consider the function $f(x) = x^2 + 1$. What is the area between the curve and the $x-$axis from 0 to 1?

Step 1: Approximate area with 1 rectangle. There are 2 ways I could do this. I could draw the rectangle so that it is an underestimate for the area. I could also draw the rectangle so that it is an overestimate. The area is given by:

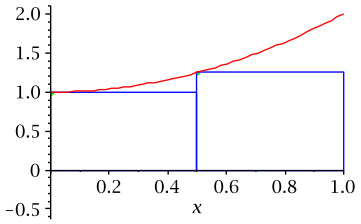


$A = 1 \cdot 1$

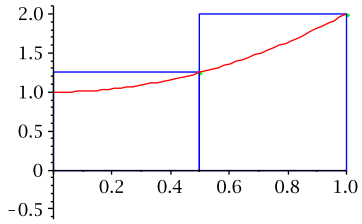


$A = 1 \cdot 2$

Step 2: Approximate with 2 rectangles. There are lots of ways I could do this. To make the process simpler, we will make all the rectangles of the same width. There are still lots of ways to put the rectangles in, so I will make another simplification. The first is called the Left-Hand Method. It says that the height of each box is the left-hand endpoint of the subinterval. The second is called the Right-Hand Method. It says that the height of each box is determined by the right-hand endpoint of each subinterval. The areas are:

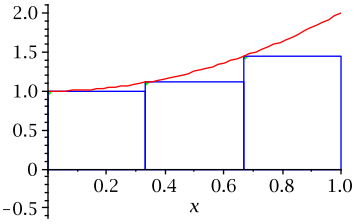


$A = 1 \cdot .5 + (\frac{1}{4} + 1) \cdot \frac{1}{2} = \frac{9}{8}$

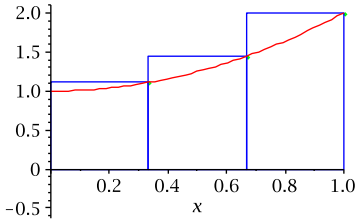


$A = (\frac{1}{4} + 1) \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{13}{8}$

Step 3: We repeat the above procedure for 3 boxes, all of equal width. The area for each method is given below:

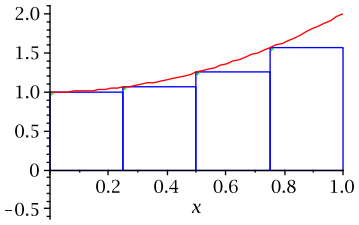


$A = 1 \cdot \frac{1}{3} + (\frac{1}{9} + 1)\frac{1}{3} + (\frac{4}{9} + 1)\frac{1}{3} = \frac{32}{27} \approx 1.185$



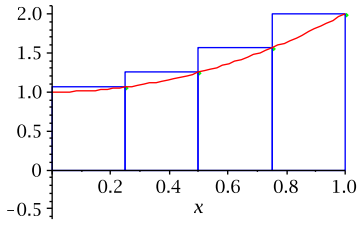
$A = (\frac{1}{9} + 1)\frac{1}{3} + (\frac{4}{9} + 1)\frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{41}{27} \approx 1.518$

Step 4: We repeat the above procedure for 4 boxes, all of equal width. The areas for each method are given by:



$A = .25(1) + .25(\frac{1}{16} + 1) + .25(\frac{1}{4} + 1) + .25(\frac{9}{16} + 1)$

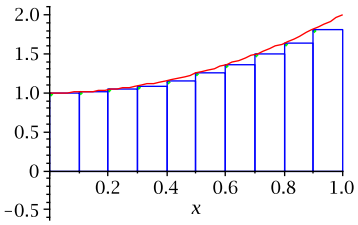
$A \approx 1.218$



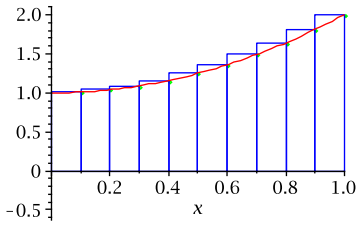
$A = .25(\frac{1}{16} + 1) + .25(\frac{1}{4} + 1) + .25(\frac{9}{16} + 1) + .25(2)$

$A \approx 1.468$

Step N: We repeat the above procedure for N boxes. Shown below is $N = 10$. What are the formulas for this ???



$A \approx 1.285$



$A \approx 1.385$

$$L_n = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{n^2}(k-1)^2 + 1\right)$$

$$R_n = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{n^2}k^2 + 1\right)$$

A good estimate of the area, is, therefore, given by the following limits.

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{n^2}k^2 + 1\right) \\
&= \lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\sum_{k=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n}\cdot n \\
&= \frac{1}{3} + 1 \\
&= \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} L_n &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{n^2}(k-1)^2 + 1\right) \\
&= \lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} (k^2 - 2k + 1) + \frac{1}{n}\sum_{k=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{2}{n^3}\frac{n(n+1)}{2} + 1 \\
&= \frac{4}{3}
\end{aligned}$$

In general, we have the following theorem about the left-hand and right-hand endpoint method. We will prove this in much greater generality in the next lecture.

Theorem 1. *Let $f(x)$ be a continuous function on a closed interval $[a,b]$. We have the following general formulas for the left and right hand methods:*

$$L_n = \sum_{k=1}^{n} \underbrace{f\left(a + \frac{b-a}{n}(k-1)\right)}_{\text{height of kth box}} \cdot \underbrace{\frac{b-a}{n}}_{\text{width of every box}}$$

$$R_n = \sum_{k=1}^{n} \underbrace{f\left(a + \frac{b-a}{n}(k)\right)}_{\text{height of kth box}} \cdot \underbrace{\frac{b-a}{n}}_{\text{width of every box}}$$

Then $\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$.

15.2. **The Riemann Integral.** To prove all of the statements we know and love about integrals, it is actually easier to consider a larger class of "methods" to compute the integral. Instead of rectangles of a fixed width, we consider boxes which have any width. Instead of the heights of the boxes determined by any endpoint, we have heights of boxes determined by any point in each subinterval.

Definition: Let $[a,b]$ be a closed interval. A partition of $[a,b]$ is a finite set of point $\{x_i : 0 \le i \le n\}$ where:

$$x_0 = a < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

The norm of the partition, $||P|| = \max\{x_i - x_{i-1} : 1 \le i \le n\}$. If P_1 and P_2 are partitions and $P_1 \subset P_2$, then P_2 is said to be a refinement of P_1.

Definition: A tagged partition, ${}^tP$, is a partition $x_0 < x_1 < \ldots < x_n$ together with a set of point $t_1,\ldots,t_n$, where $x_{i-1} \le t_i \le x_i$. A tagged partition is often written ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \le i \le n\}$.

Definition: Given a partition ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \le i \le n\}$, the Riemann sum is given by:

$$S(f, {}^tP) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Definition:(The Riemann Integral) We say that a function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if there is a real number L such that for all $\varepsilon > 0$, there is a $\delta > 0$ such that if ${}^tP$ is a tagged partition of $[a,b]$ with $||P|| < \delta$, then:

$$|S(f, {}^tP) - L| < \varepsilon$$

The number L, if it exists, is called the Riemann integral of f on $[a,b]$ and is denoted:

$$\int_a^b f(x)dx$$

The symbol $\int$ is called the integral sign. The symbol dx is called the differential. The number a is called the lower bound and the number b is called the upper bound.

Example: Show that the function defined below is Riemann integrable on $[0,2]$:

$$f(x) = \begin{cases} \pi & x \in (0,2] \\ 0 & x = 0 \end{cases}$$

A good guess is that the Riemann integral is equal to 2π. If P is a partition, the only subinterval of concern is the one that contains x_1. The rest of the Riemann sum is just $\pi(x_n - x_1) = \pi(2 - x_1)$. On the interval $[0, x_1]$, we will either have that the tag is at 0 or it is not at 0. If it is not at 0, the Riemann sum $S(f, {}^tP)$ is exactly 2π. If it is at 0, the Riemann sum is:

$$S(f, {}^tP) = \pi(2 - x_1) + 0$$

$$|S(f,^t P) - 2\pi| = |\pi x_1| = \pi|x_1 - x_0| < \pi\delta$$

Therefore, we need $\delta = \varepsilon/\pi$. Thus, if δ is chosen in this way, $|S(f,^t P) - 2\pi| < \varepsilon$. Hence, f is Riemann integrable.

Example: Show that the Dirichlet function is not Riemann integrable on $[0, 1]$. We will show that for all $\delta > 0$, there exist tagged partitions of norm $< \delta$ such that $S(f,^t P_1) = 1$ and $S(f,^t P_2) = 0$. Indeed, if P is any partition with $||P|| < \delta$, choose a collection r_i of rational numbers such that r_i is in the $i - th$ subinterval. Choose a collection of irrational numbers such that t_i is in the ith subinterval. It is easy to see that these tagged partitions are the ones that we want. Thus, the Dirichlet function is not Riemann integrable on $[0, 1]$.

Theorem 2. *Suppose that f and g are Riemann integrable on $[a, b]$ and k is a constant. The following statements are valid.*

1. *kf is Riemann integrable with $\int_a^b f(x)dx = k\int_a^b f(x)dx$.*
2. *$f + g$ is Riemann integrable with $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$*
3. *If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.*
4. *If $|f(x)| \leq M$ for all $x \in [a, b]$, then $\left|\int_a^b f(x)dx\right| \leq M(b - a)$.*

Proof. We prove (2) and (3) only. For (2), we see that for any tagged partition ${}^tP$:

$$S(f + g,^t P) = S(f,^t P) + S(g,^t P)$$

Let $\varepsilon > 0$ be given. Choose δ_1 such that if $||P|| < \delta_1$, then $|S(f,^t P) - \int_a^b f(x)dx| < \varepsilon/2$. Choose δ_2 such that if $||P|| < \delta_2$, then $|S(g,^t P) - \int_a^b f(x)dx| < \varepsilon/2$. Then if $||P|| < \delta = \min\{\delta_1, \delta_2\}$:

$$\begin{aligned}
\left|S(f + g,^t P) - \left(\int_a^b f(x)dx + \int_a^b g(x)dx\right)\right| &= \left|S(f,^t P) + S(g,^t P) - \int_a^b f(x)dx - \int_a^b g(x)dx\right| \\
&\leq \left|S(f,^t P) - \int_a^b f(x)dx\right| + \left|S(g,^t P) - \int_a^b g(x)dx\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

For (3), it is sufficient to show that if $f \geq 0$, then:

$$\int_a^b f(x)dx \geq 0$$

The actual statement then follows from (1) and (2). However, this is almost obvious because in this case $S(f,^t P) \geq 0$ for all partitions. Hence, the Riemann integral must also be nonnegative. The class completed the proof of this theorem as an in-class exercise. □

LECTURE 16: CONDITIONS FOR INTEGRABILITY

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon and "Analysis With an Introduction to Proof" by Steven R. Lay.

16.1. **Riemann Integrable Implies Bounded.** In many texts, all Riemann integrable functions are assumed to be bounded as part of the definition. The definition given in our text does not require the function to be bounded. However, it turns out that every Riemann integrable function is bounded.

Theorem 1. *If f is Riemann integrable on $[a,b]$, then f is bounded on $[a,b]$.*

Proof modified from Gordon text. The idea is that you can estimate the size of f on small intervals of equal size using Riemann sums on sufficiently small partitions. Choose $\delta > 0$ such that if $||P|| < \delta$, then for any tagging of P, we have:

$$\left| \int_a^b f(x)\,dx - S(f,^t P) \right| < 1/2$$

There is an integer q so that $\beta = (b-a)/q < \delta$. For $0 \leq j \leq q$, define $x_j = a + j\beta$. Define ${}^tP_0 = \{(x_j, [x_{j-1}, x_j]) : 1 \leq j \leq q\}$.

Let $x \in [a,b]$. There is a j so that $x_{j-1} \leq x \leq x_j$. Set ${}^tP = ({}^tP_0 \backslash \{(x_j, [x_{j-1}, x_j])\}) \cup \{(x, [x_{j-1}, x_j)\}$.

$$\begin{aligned} |S(f,^t, P) - S(f,^t P_0)| &= |f(x)(x_j - x_{j-1}) - f(x_j)(x_j - x_{j-1})| \\ &= |f(x) - f(x_j)| \cdot \beta \end{aligned}$$

Since $||P|| = \beta < \delta$, we also have:

$$\begin{aligned} |S(f,^t P) - S(f,^t P_0)| &= \left| S(f,^t P) - \int_a^b f(x)\,dx + \int_a^b f(x)\,dx - S(f,^t P_0) \right| \\ &< \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned}$$

Therefore, by the reverse triangle inequality, we have:

$$|f(x)| \leq \frac{1}{\beta} + |f(x_j)|.$$

Let $M = \max\{\frac{1}{\beta} + |f(x_j)| : 1 \leq j \leq q\}$. Therefore, if $z \in [a,b]$, we have $|f(z)| \leq M$. □

16.2. **The Darboux Integral.** At this point we will depart from the Gordon text and present an argument originally due to Darboux. We will follow the exposition given in the Lay text. The key point is that Riemann integrable functions are bounded, so we can make bound all possible Riemann sums from above and below.

Definition 2. *Let f be a bounded function on $[a,b]$. Let $P = \{x_i : 0 \leq i \leq n\}$ be a partition of $[a,b]$. Define:*

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \\ L(f,P) &= \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \end{aligned}$$

$U(f,P)$ is called the ***upper sum of f with respect to P.*** *$L(f,P)$ is called the* ***lower sum of f with respect to P.***

$$\begin{aligned} U(f) &= \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} \\ L(f) &= \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \end{aligned}$$

$U(f)$ is called the ***upper integral.*** *$L(f)$ is called the* ***lower integral.*** *If $L(f) = U(f)$, then f is said to be* ***Darboux integrable.***

We will show later that a bounded function is Darboux integrable if and only if it is Riemann integrable. Some things are easier to do with the Darboux integral definition whereas other things are easier to do with the Riemann integral definition. We will simply talk about functions being "integrable". We will use the easiest definition for the situation at hand.

Lemma 3. *Let f be a bounded function on $[a,b]$ and P_0 a partition. Let P_1 be a refinement of P_0. Then:*

$$L(f,P_0) \leq L(f,P_1) \leq U(f,P_1) \leq U(f,P_0)$$

Proof. It is clear from the definition that $L(f,P_i) \leq U(f,P_i)$ for $i = 0,1$. Suppose that P_1 has exactly one more point y than P_0, so that there is a j such that $x_{j-1} < y < x_j$. Define:

$$\begin{aligned} M_0 &= \sup\{f(x) : x_{j-1} < x < x_j\} \\ M_1 &= \sup\{f(x) : x_{j-1} < x < y\} \\ M_2 &= \sup\{f(x) : y < x < x_j\} \end{aligned}$$

It follows that:

$$\begin{aligned} M_0(x_j - x_{j-1}) &= M_0(x_j - y) + M_0(y - x_{j-1}) \\ &\geq M_2(x_j - y) + M_1(y - x_{j-1}) \end{aligned}$$

Thus, $U(f,P_0) \geq U(f,P_1)$. If P_1 has more than one more point than P_0, the result is obtained by adding one point at a time to P_0 until you get to the partition P_1. The inequality $L(f,P_0) \leq L(f,P_1)$. □

Lemma 4. *If f is a bounded function, then $L(f) \leq U(f)$.*

Proof. It follows from the above that $U(f)$ is an upper bound for $L(f,P)$ for every partition P of $[a,b]$. □

Lemma 5. *Let f be a bounded function on $[a,b]$. Then f is (Darboux) integrable if and only if for every $\varepsilon > 0$, there is a partition P of $[a,b]$ such that:*

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof modified from Lay text. Suppose that $L(f) = U(f)$ and let $\varepsilon > 0$ be given. By definition of supremum, we have that there is a partition P_1 such that $L(f,P_1) > L(f) - \varepsilon/2$. By definition of infimum, there is a partition P_2 such that $U(f,P_2) < U(f) + \varepsilon/2$. Then $P = P_1 \cup P_2$ is a refinement of P_1 and P_2. It follows that:

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,P_2) - L(f,P_1) \\ &< U(f) + \frac{\varepsilon}{2} - L(f) + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

For the reverse direction, suppose that $L(f) < U(f)$. Let $\varepsilon = (U(f) - L(f))/2$. Let P be the given partition for this ε. Thus,

$$U(f) \leq U(f,P) < L(f,P) + \varepsilon \leq L(f) + \frac{U(f) - L(f)}{2} = \frac{U(f) + L(f)}{2} \leq U(f).$$

This gives the contradiction that $U(f) < U(f)$. Thus we must have that $U(f) = L(f)$. Hence f is (Darboux) integrable. □

16.3. **Continuous Functions.** It can now be proved that continuous functions are (Darboux) integrable. We will show later that Darboux integrable functions are Riemann integrable and vice versa. Hence, continuous functions are also Riemann integrable.

Theorem 6. *If f is a continuous function on $[a,b]$, then f is (Darboux) integrable on $[a,b]$.*

Proof modified from Lay text. Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$. Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/(b-a)$.

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that $||P|| < \delta$. By the EVT, for each j, there is $c_j \in [x_{j-1}, x_j]$ such that $f(c_j) \leq f(x)$ for all $x \in [x_{j-1}, x_j]$. Similarly, there is a $d_j \in [x_{j-1}, x_j]$ such that $f(d_j) \geq f(x)$ for all $x \in [x_{j-1}, x_j]$. Using the definition of $U(f,P)$ and $L(f,P)$, we have that:

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{j=1}^{n} f(d_j)(x_j - x_{j-1}) - \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} (f(d_j) - f(c_j))(x_j - x_{j-1}) \\
&< \sum_{j=1}^{n} \frac{\varepsilon}{(b-a)}(x_j - x_{j-1}) \\
&= \frac{\varepsilon}{(b-a)} \sum_{j=1}^{n} (x_j - x_{j-1}) \\
&= \frac{\varepsilon}{(b-a)}(b-a) \\
&= \varepsilon
\end{aligned}$$

By the previous lemma, we have that f is (Darboux) integrable. □

Examples:(from Lay text) The function defined to be $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ is integrable on $[0,1]$. The modified Dirichlet function in integrable on $[0,1]$.

$$g(x) = \begin{cases} \frac{1}{n}, & x = \frac{m}{n}, \gcd(m,n) = 1 \\ 0 & x \text{ is irrational} \end{cases}$$

LECTURE 17: THE FUNDAMENTAL THEOREM OF CALCULUS

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

17.1. **Introduction.** In this lecture we will prove a general form of the Fundamental Theorem of Calculus. We have proved that Riemann integrable functions are bounded. We have proved that continuous functions on a closed interval are Darboux integrable. We have also proved some of the familiar properties of integrals from calculus using the Riemann definition of integrability. Now, the FTC from calculus looks typically something like this.

Theorem 1 (FTC for continuous functions)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous.*

(1) *Define $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a, b]$, differentiable on (a, b) and $F' = f$.*
(2) *If F is any antiderivative of f, then $\int_a^b f(t)\,dt = F(b) - F(a)$.*

The general version has weaker hypotheses. It is assumed only that the function f is Riemann integrable. To recover the calculus version from the general version, it is thus necessary to prove that continuous functions are Riemann integrable. Our subsequent lecture will prove that a bounded function is Darboux integrable iff it is Riemann integrable.

The "Darboux iff Riemann" theorem is important for the FTC in second way. Recall from calculus that if a,b,c are real numbers and f is Riemann integrable on an interval containing them, then we have:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Our proof of this fact will use *both* the Riemann and Darboux interpretations of the integral. As we will see, the FTC uses this fact from calculus several times. We will delay the proof of this result until our "Technical Theorems" lecture.

Recall also that we have the following two conventions, for $a < b$:

$$\int_a^a f(x)\,dx = 0, \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

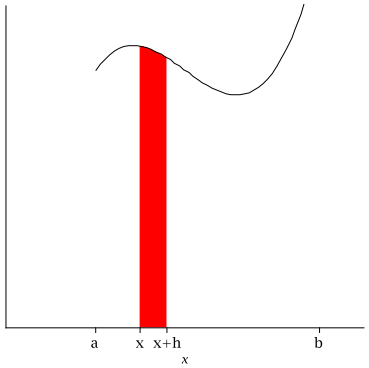
17.2. **Statement of FTC and "Picture Proof".** The general form of the fundamental theorem of calculus is given as follows.

Theorem 2 (The Fundamental Theorem of Calculus)**.** *Let f be a Riemann integrable function on $[a, b]$. Define a function $F : [a, b] \to \mathbb{R}$ by:*

$$F(x) = \int_a^x f(t)dt.$$

(1) *F is continuous on $[a, b]$. If f is continuous at $x \in (a, b)$, then F is differentiable at x and $F'(x) = f(x)$.*
(2) *If H is any antiderivative of f in $[a, b]$, then $\int_a^b f(t)\,dt = H(b) - H(a)$.*

We present in this section a "picture proof".



$$f(x) \cdot h \approx \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$f(x) \approx \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$\begin{aligned} f(x) &= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\ &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= F'(x) \end{aligned}$$

17.3. **Formal Proof of FTC.** We will prove each part individually. Note the places where Riemann integrability is used and the places where Darboux integrability is used.

17.3.1. *FTC Part I: Differentiation of the Integral.* We first prove that F is uniformly continuous on $[a,b]$. Since f is Riemann integrable, we have that f is bounded on $[a,b]$. Let $M>0$ be a real number such that $|f(x)| \le M$ for all $x \in [a,b]$. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/M$. Then we have:

$$\begin{aligned} |F(x)-F(y)| &= \left|\int_a^x f(t)\,dt - \int_a^y f(t)\,dt\right| \\ &= \left|\int_x^a f(t)\,dt + \int_a^y f(t)\,dt\right| \\ &= \left|\int_x^y f(t)\,dt\right| \\ &\le M|y-x| \\ &< M\delta \\ &= \varepsilon \end{aligned}$$

Now suppose that f is continuous at $x \in (a,b)$. It will be shown that the limit defining $F'(x)$ is defined and equal to $f(x)$. In particular, we want to show that if $\varepsilon > 0$ is given, there is a $\delta > 0$ such that if $|z-x| < \delta$, then:

$$\left|\frac{F(z)-F(x)}{z-x} - f(x)\right| < \varepsilon.$$

First note that $f(x)$ can be be written as:

$$f(x) = \frac{1}{z-x} f(x)(z-x) = \frac{1}{z-x}\int_x^z f(x)\,dt$$

Also, we have that $\frac{F(z)-F(x)}{z-x}$ can be written as a single integral using the properties of the integral.

$$\frac{F(z)-F(x)}{z-x} = \frac{1}{z-x}\left(\int_a^z f(t)\,dt - \int_a^x f(t)\,dt\right) = \frac{1}{z-x}\int_x^z f(t)\,dt.$$

Putting these two facts together gives the following:

$$\left|\frac{F(z)-F(x)}{z-x} - f(x)\right| = \frac{1}{|z-x|}\left|\int_x^z f(t) - f(x)\,dt\right|$$

Now let $\varepsilon > 0$ be given. Since f is continuous at x, there is a $\delta > 0$ such that if $|t-x| < \delta$, then $|f(t)-f(x)| < \varepsilon$. This means that the function $f(t)-f(x)$ is bounded on the interval between x and z by ε. Again using the properties of the integral, we obtain that:

$$\begin{aligned}\left|\frac{F(z)-F(x)}{z-x} - f(x)\right| &= \frac{1}{|z-x|}\left|\int_x^z f(t) - f(x)\,dt\right| \\ &\leq \frac{1}{|z-x|}\varepsilon|z-x| \\ &= \varepsilon\end{aligned}$$

Thus, F is differentiable at x and $F'(x) = f(x)$.

17.3.2. *FTC Part II: "$F(b)-F(a)$".* Suppose that H is an antiderivative of f on $[a,b]$. In other words, suppose that $H:[a,b]\to\mathbb{R}$ is a continuous function such that $H'(x) = f(x)$ for all $x \in (a,b)$. Let $P = \{x_0,\ldots,x_n\}$ be a partition of $[a,b]$. On each interval $[x_{j-1},x_j]$ we have that H satisfies the hypotheses of the Mean Value theorem. Hence, there is a $c_j \in (x_{j-1},x_j)$ such that:

$$f(c_j) = H'(c_j) = \frac{H(x_j)-H(x_{j-1})}{x_j - x_{j-1}}$$

Define the tagged partition ${}^tP = \{(c_j,[x_{j-1},x_j])\}$. It follows that:

$$\begin{aligned}S(f,{}^tP) &= \sum_{j=1}^n f(c_j)(x_j - x_{j-1}) \\ &= \sum_{j=1}^n H(x_j) - H(x_{j-1}) \\ &= H(x_n) - H(x_0) \\ &= H(b) - H(a)\end{aligned}$$

Let $\varepsilon > 0$ be given. Now, since f is (Darboux) integrable, we have that $L(f) = U(f)$ and there is a partition P of $[a,b]$ such that:

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2}$$

Now, $L(f) \in (L(f,P),U(f,P))$. Since $L(f,P) \leq S(f,{}^tP) \leq U(f,{}^tP)$, we have that $H(b)-H(a) \in (L(f,P),U(f,P))$. Thus, $|L(f) - (H(b)-H(a))| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $L(f) = H(b)-H(a)$.

17.4. **Examples involving FTC:.** Let's revisit some standard calculus problems.

Example: Compute the derivative of the following function:

$$F(x) = \int_1^{x^2} \sin(t^2)\,dt$$

Example: In terms of simple examples, the FTC is easy to verify. Consider $f(x) = 3x+4$ on $[0,3]$. For $t \in [0,3]$, the area between the $x-$axis and the curve from 0 to t is given by the function:

$$A(t) = 4t + \frac{1}{2}t\cdot 3t$$

It is easy to see that $A(x)$ is an antiderivative of $f(x)$. That is all that FTCI says in this case. Also, we see that the area under the curve from 0 to 3 is precisely $A(3) - A(0)$. This is exactly what FTCII says!

Example: Compute the following integral:

$$\int_0^{\pi} \sin(t)\, dt = -\cos(t)|_0^{\pi} = -\cos(\pi) + \cos(0) = 2$$

Example: Not all derivatives are Riemann integrable. It is not true that $\int_a^b f'(t)\, dt = f(b) - f(a)$. This only works when f' is integrable (for example, when f' is continuous or f is C^1). Consider the following function.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is differentiable and its derivative is given by:

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Let $x_n = 1/\sqrt{2\pi n}$ for $n \in \mathbb{N}$. $f'(x_n) = -2\sqrt{2\pi n}$. This implies that f' is not bounded on $[0, 1]$. Hence, f' is not Riemann integrable on $[0, 1]$.

LECTURE 18: TECHNICAL INTEGRATION THEOREMS

These are notes taken from the book "Real Analysis: A First Course", by Russel A. Gordon.

18.1. **Darboux integrable iff Riemann integrable.** This section provides a proof that a bounded function is Darboux integrable iff it is Riemann integrable. The proof requires a number of lemmas. We will be roughly following the proofs in the Gordon text but using the more standard notation that is given in the Lay text.

Lemma 1. *Let f be a bounded function on $[a,b]$ and $P_1 \subseteq P_2$ be partitions. Then:*

$$U(f,P_2) - L(f,P_2) \leq U(f,P_1) - L(f,P_1)$$

Proof. This follows immediately from the "finer sup goes down" lemma. □

Lemma 2 (Cauchy Criterion for Riemann Integrability)**.** *Let f be a bounded function on $[a,b]$. Then f is Riemann integrable if and only if for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all tagged partitions ${}^tP_1$ and ${}^tP_2$ satisfying $||{}^tP_1|| < \delta$, $||{}^tP_2|| < \delta$, we have that:*

$$|S(f,{}^tP_1) - S(f,{}^tP_2)| < \varepsilon$$

Proof. We will prove the reverse direction first. The idea behind the proof is to use the given condition to construct a Cauchy sequence of Riemann sums, the norm of whose partitions goes to zero. Since every Cauchy sequence of real numbers converges, we have a limiting value L. It is then shown that L satisfies the definition of the Riemann integral.

Construction of the Cauchy sequence: Let δ_n be the number given from the hypothesis with $\varepsilon = 1/n$. The sequence can be assumed to be decreasing, because if $\delta_{n+1} > \delta_n$ for some n, then we just redefine $\delta_{n+1} = \delta_n$. Now let ${}^tP_n$ a tagged partition of $[a,b]$ satisfying $||{}^tP_n|| < \delta_n$.

Let $\varepsilon > 0$ be given. Choose N so large that $1/N < \varepsilon$. Then if $j,k \geq N$, we have that $||P_j|| < \delta_j \leq \delta_N$ and $||P_k|| < \delta_k \leq \delta_N$. Hence, it follows that:

$$|S(f,{}^tP_j) - S(f,{}^tP_k)| < \frac{1}{N} = \varepsilon.$$

Thus, $\{S(f,{}^tP_n)\}$ is a Cauchy sequence. Let L be the limit of this sequence of REAL NUMBERS.

L is the Riemann integral for f on $[a,b]$: Let $\varepsilon > 0$ be given. Since $S(f,{}^tP_n) \to L$, there is an $N > 0$ such that if $n \geq N$, $|S(f,{}^tP_n) - L| < \varepsilon/2$. We may also assume that N is so large that $1/N < \varepsilon/2$. Define $\delta = \delta_N$, where δ_N is as above. Let ${}^tP$ be a tagged partition with $||{}^tP|| < \delta$. Thus, we have:

$$\begin{aligned} |S(f,{}^tP) - L| &= |S(f,{}^tP) - S(f,{}^tP_n) + S(f,{}^tP_n) - L| \\ &\leq |S(f,{}^tP) - S(f,{}^tP_n)| + |S(f,{}^tP_n) - L| \\ &< \frac{1}{N} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Thus, f is Riemann integrable.

For the forward direction, let L be the value of the Riemann integral. Let $\varepsilon > 0$ and let δ be the number given by the definition of the Riemann integral for $\varepsilon/2$. Apply the add-and-subtract-the-same-thing-trick with L. □

Lemma 3. *Let f be a bounded function on $[a,b]$. Let $P_1 \subseteq P_2$ be partitions of $[a,b]$ and ${}^tP_1$, ${}^tP_2$ taggings of P_1 and P_2, respectively. Then:*

$$|S(f,{}^tP_1) - S(f,{}^tP_2)| < U(f,P_1) - L(f,P_1).$$

Proof. It is always true that $L(f,P_1) \leq S(f,{}^tP_1) \leq U(f,P_1)$. From the "finer sup goes down" lemma, we have that $L(f,P_1) \leq L(f,P_2) \leq S(f,{}^tP_2) \leq U(f,P_2) \leq U(f,P_1)$. Putting these two facts together gives:

$$-U(f,P_1) + L(f,P_1) \leq S(f,{}^tP_1) - S(f,{}^tP_2) \leq U(f,P_1) - L(f,P_1)$$

This proves the lemma. Note that our proof of the "finer soup goes down" makes this much easier than what you see in the text. □

Theorem 4. *Let f be a bounded function on $[a,b]$. Then f is Darboux integrable if and only if it is Riemann integrable.*

Proof. Suppose that f is Darboux integrable on $[a,b]$. We will prove that f is Riemann integrable using the Cauchy criterion. Let $M > 1$ be a bound for f on $[a,b]$.

Let $0 < \varepsilon < 1$ be given. Then there is a partition Q of $[a,b]$ such that:

$$U(f,Q) - L(f,Q) < \frac{\varepsilon}{4}$$

There is a refinement $Q_\varepsilon = \{z_0, \ldots, z_N\}$ of Q such that $||Q_\varepsilon|| < 1$. Since Q_ε is a refinement, we have that:

$$U(f,Q_\varepsilon) - L(f,Q_\varepsilon) \leq U(f,Q) - L(f,Q) < \frac{\varepsilon}{4}$$

Fix a tagging ${}^tQ_\varepsilon$. Define d to be the length of the smallest subinterval of Q_ε. Note that $d \leq ||Q_\varepsilon|| < 1$. Define $\delta = d\varepsilon/(8MN)$. Note that $\delta < d$. We will compare all other tagged partitions to this tagged partition.

Let ${}^tP = \{(t_i, [x_{i-1}, x_i])\}$ be any tagged partition satisfying $||P|| < \delta$. The longest subinterval of P has length less than δ and no subinterval of Q has length shorter than d. If any subinterval of P contains two points of Q_ε, then its length would be $\geq d$. This gives the contradiction that $\delta \geq d$. Hence, we must have that each subinterval of P contains at most one element of Q_ε. Define (as in your text):

$$A := \{i : (x_{i-1}, x_i) \cap Q_\varepsilon \neq \emptyset\}$$

If $i \in A$, define z_{k_i} by $(x_{i-1}, x_i) \cap Q_\varepsilon = \{z_{k_i}\}$. Note that $|A| < N$, since $k_i \neq 0, N$ for any i.

The next task is to use P to construct a refinement of Q_ε. Define $Q_0 = P \cup Q_\varepsilon$. The tagging of Q_0 is determined by A. If $i \notin A$, then set the tag of the subinterval $[x_{i-1}, x_i]$ to be t_i (Q_ε does not split this subinterval). If $i \in A$, then set z_{k_i} to be the tag of BOTH subintervals $[x_{i-1}, z_{k_i}]$ and $[z_{k_i}, x_i]$. This defines a tagging of ${}^tQ_0$.

As a last observation, note that if $i \in A$, we have:

$$f(z_{k_i})(x_i - x_{i-1}) = f(z_{k_i})(x_i - z_{k_i}) + f(z_{k_i})(z_{k_i} - x_{i-1})$$

The right shows the contribution to the Riemann sum $S(f,{}^t Q_0)$. The left shows that these two terms collapse to a single term. Finally, we are ready to estimate.

$$\begin{aligned}
|S(f,{}^t P) - S(f,{}^t Q_\varepsilon)| &\leq \left|S(f,{}^t P) - S(f,{}^t Q_0)| + |S(f,{}^t Q_0) - S(f,{}^t Q_\varepsilon)\right| \\
&= \left|\sum_{i\in A}(f(t_i) - f(z_{k_i})(x_i - x_{i-1})\right| + |S(f,{}^t Q_0) - S(f,{}^t Q_\varepsilon)| \\
&\leq \sum_{i\in A}(|f(t_i)| + |f(z_{k_i})|)\delta + U(f,Q_\varepsilon) - L(f,Q_\varepsilon) \\
&< 2MN\delta + \frac{\varepsilon}{4} \\
&= 2MNd\frac{\varepsilon}{8MN} + \frac{\varepsilon}{4} \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \frac{\varepsilon}{2}
\end{aligned}$$

To finish the proof, let ${}^t P_1$ and ${}^t P_2$ be any partitions of $[a,b]$ of norm less than δ. Then:

$$\begin{aligned}
|S(f,{}^t P_1) - S(f,{}^t P_2)| &\leq |S(f,{}^t P_1) - S(f,{}^t Q_\varepsilon)| + |S(f,{}^t Q_\varepsilon) - S(f,{}^t P_)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus, f is Riemann integrable by the Cauchy criterion.

Now suppose that f is Riemann integrable. We sketch the argument in this case. Let L be the Riemann integral and δ be the number obtained from the definition of the Riemann integral for $\varepsilon/2$. Then if $||P|| < \delta$, we have that $S(f,{}^t P) \in (L - \varepsilon/2, L + \varepsilon/2)$ for all taggings ${}^t P$ of P. It follows that $U(f,P) \in (L - \varepsilon/2, L + \varepsilon/2)$ and $L(f,P) \in (L - \varepsilon/2, L + \varepsilon/2)$. Then there is a partition P such that $U(f,P) - L(f,P) < \varepsilon$. This is an equivalent condition for Darboux integrability. □

Corollary 5. *Let f be a bounded function on $[a,b]$. If f is integrable, then the value of the Riemann integral is equal to the value of the Darboux integral.*

Proof. From the proof that Riemann integrability implies Darboux integrability, we have that $L(f) = U(f) = L = \int_a^b f(x)\,dx$. □

18.2. **An Important Property of Integrals.** We have exactly one more thing to establish before we can be convinced of the general form of the FTC beyond any doubt. It too needs a lemma first.

Lemma 6. *If $f : [a,b] \to \mathbb{R}$ is (Riemann) integrable, and $\{{}^t P_n\}$ is a sequence of tagged partitions of $[a,b]$ such that $||{}^t P_n|| \to 0$, then $\{S(f,{}^t P_n)\} \to \int_a^b f(x)dx$.*

Proof. Let $\varepsilon > 0$ be given. Since f is Riemann integrable, there is a $\delta > 0$ such that if $||{}^t P|| < \delta$, then:

$$\left|S(f,{}^t P) - \int_a^b f(x)dx\right| < \varepsilon$$

Since $||{}^t P_n|| \to 0$, there is an $N \geq 0$ such that if $n \geq N$, $||{}^t P_n|| < \delta$. Hence, if $n \geq N$, we have:

$$\left|S(f,{}^t P_n) - \int_a^b f(x)dx\right| < \varepsilon$$

Thus, $\{S(f,{}^t P_n)\} \to \int_a^b f(x)dx$, by the $\varepsilon - N$ definition of a convergent sequence. □

As an immediate consequence, we get the following theorem that we use to define the definite integral in our Calculus I course.

Theorem 7. *Let $f(x)$ be a continuous function on a closed interval $[a,b]$. We have the following general formulas for the left and right hand methods:*

$$L_n = \sum_{k=1}^{n} \underbrace{f\left(a + \frac{b-a}{n}(k-1)\right)}_{\text{height of kth box}} \cdot \underbrace{\frac{b-a}{n}}_{\text{width of every box}}, R_n = \sum_{k=1}^{n} \underbrace{f\left(a + \frac{b-a}{n}(k)\right)}_{\text{height of kth box}} \cdot \underbrace{\frac{b-a}{n}}_{\text{width of every box}}$$

Then $\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$.

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$.*

1. *For every subinterval $[u,v] \subseteq [a,b]$, f is integrable on $[u,v]$.*
2. *If f is integrable on $[a,c]$ and on $[c,b]$, then f is integrable on $[a,b]$ and,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof. Let $\varepsilon > 0$ be given. Since f is (Darboux) integrable, there is a partition P such that:

$$U(f,P) - L(f,P) < \varepsilon.$$

Let $P_0 = P \cup \{u,v\}$. Since P_0 is a refinement of P, we have that:

$$U(f,P_0) - L(f,P_0) \leq U(f,P) - L(f,P) < \varepsilon$$

Let $P_1 = P_0 \cap [u,v]$. Then P_1 is a finite set of points. Note that

$$U(f,P_0) - L(f,P_0) = \sum_{k=1}^{n} \omega_k (x_k - x_{k-1})$$

where $\omega_k = \sup\{f(x) : x_{k-1} \leq x \leq x_k\} - \inf\{f(x) x_{k-1} \leq x \leq x_k\} \geq 0$. Then $U(f,P_1) - L(f,P_1)$ is a sum of a finite number of the terms from this sum. Hence:

$$U(f,P_1) - L(f,P_1) \leq U(f,P_0) - L(f,P_0) < \varepsilon.$$

Thus f is (Darboux) integrable on $[u,v]$.

Let $P_{[a,c]}$ be a partition of $[a,c]$ such that $U(f,P_{[a,c]}) - L(f,P_{[a,c]}) < \varepsilon/2$. Let $P_{[c,b]}$ be a partition of $[c,b]$ such that $U(f,P_{[c,b]}) - L(f,P_{[c,b]}) < \varepsilon/2$. Then $P_{[a,c]} \cup P_{[c,b]}$ is a partition of $[a,b]$ and:

$$\begin{aligned} U(f, P_{[a,c]} \cup P_{[c,b]}) - L(f, P_{[a,c]} \cup P_{[c,b]}) &= U(f,P_{[a,c]}) + U(f,P_{[c,b]}) - L(f,P_{[a,c]}) - L(f,P_{[c,b]}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

This implies that f is (Darboux) integrable on $[a,b]$.

Suppose that we have a sequence $\{{}^tP_n\}$ of tagged partitions of $[a,c]$ such that $||{}^tP_n|| \to 0$ and a sequence $||{}^tQ_n||$ of tagged partitions of $[c,b]$ such that $||{}^tQ_n|| \to 0$. For each n, let ${}^tR_n = {}^t P_n \cup {}^t Q_n$. Then $||{}^tR_n|| \to 0$ and we have by several applications of the previous lemma that:

$$\begin{aligned} \int_a^b f(t)\,dt &= \lim_{n\to\infty} S(f, {}^t R_n) \\ &= \lim_{n\to\infty} S(f, {}^t P_n) + S(f, {}^t Q_n) \\ &= \lim_{n\to\infty} S(f, {}^t P_n) + \lim_{n\to\infty} S(f, {}^t Q_n) \\ &= \int_a^c f(t)\,dt + \int_c^b f(t)\,dt \end{aligned}$$

□